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Special geometry of euclidean supersymmetry II. Hypermultiplets and the c-map

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Abstract: We construct two new versions of the c-map which allow us to obtain the target manifolds of hypermultiplets in euclidean theories with rigid $\mathcal{N} = 2$ supersymmetry. While the minkowskian para-c-map is obtained by dimensional reduction of the minkowskian vector multiplet lagrangian over time, the euclidean para-c-map corresponds to the dimensional reduction of the euclidean vector multiplet lagrangian. In both cases the resulting hypermultiplet target spaces are para-hyper-Kähler manifolds. We review and prove the relevant results of para-complex and para-hypercomplex geometry. In particular, we give a second, purely geometrical construction of both c-maps, by proving that the cotangent bundle $N = T^*M$ of any affine special (para-)Kähler manifold M is para-hyper-Kähler.

Keywords: Extended Supersymmetry, Differential and Algebraic Geometry.

Contents

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1. Introduction, summary and outlook

1.1 Introduction

In this paper we continue the investigation of the geometrical structures of euclidean supersymmetric theories initiated in [1]. As explained there in more detail, the main motivation of our work is to obtain a better understanding of instantons in supersymmetric string theory compactifications and their rigidly supersymmetric field theory limits. Moreover, since solitons and instantons are mutually related by dimensional reduction over time, instantons can be used as the starting point for the systematic construction of solitonic solutions, such as black holes, black branes and domain walls [2]. One also expects to be able to generate cosmological solutions of the type II* string theories, which are related to the standard type II string theories by time-like T-duality [3, 4]. Since the target space geometries of the scalar sigma models appearing in euclidean supersymmetric theories differ from those of the minkowskian theories [5 − 11], we were lead to the question what characterizes these geometries. For theories with more than 8 supercharges the scalar geometry is completely fixed by the matter content, while for theories with 8 or less supercharges the geometry is restricted, but not unique. The boundary case of 8 supercharges is particularly interesting, because the resulting theories have a rich structure while they (or at least certain aspects of them) can be treated exactly. For space-times with minkowskian signature, the corresponding geometries are known as the 'special geometries.'[1] More precisely, for locally supersymmetric theories the manifolds spanned by the scalars of vector multiplets are 'projective very special real manifolds' [13 − 15] in five dimensions and 'projective special Kähler manifolds' [16 − 19] in four dimensions, while the scalar manifolds of hypermultiplets are quaternion-Kähler [20, 21]. For rigidly supersymmetric theories the manifolds spanned by the scalars of vector multiplets are 'affine very special real manifolds' [1] in five dimensions and 'affine special Kähler manifolds' [22, 23, 18, 19] in four dimensions, while the scalar manifolds of hypermultiplets are hyper-Kähler manifolds [24] in all dimensions ≤ 6.

In [1] we investigated vector multiplets in euclidean four-dimensional space. We introduced the notion of an 'affine special para-Kähler manifold' and showed that these are the target manifolds of rigid euclidean vector multiplets. The basic difference between the scalar geometries of minkowskian and euclidean vector multiplets is that the complex structure I, $I^2 = -\mathbb{1}$ is replaced by a para-complex structure J, $J^2 = \mathbb{1}$ (with equal rank of the eigenspaces for the eigenvalues ± 1). As discussed in [1], all the relevant notions of complex geometry, such has hermitean, Kähler and special Kähler, have their precise analogues in para-complex geometry.

The purpose of this paper is to identify the scalar geometry of euclidean hypermultiplets in rigidly supersymmetric theories. Since the scalar geometry of hypermultiplets does not change under dimensional reduction, this geometry is the same for all dimensions where hypermultiplets exist, i.e., for dimensions ≤ 6. Our main result is that the scalar manifolds of euclidean hypermultiplets are para-hyper-Kähler manifolds. The precise definition will be given later, but let us already characterize them heuristically as para-complex analogues of hyper-Kähler manifolds: hyper-Kähler manifolds have three complex structures, which satisfy the quaternionic algebra under multiplication, while para-hyper-Kähler manifolds have two para-complex structures and one complex structure, which satisfy the so-called para-quaternionic algebra. Thus the resulting picture is very similar to the one we found for vector multiplets.

The most convenient tool to find these euclidean hypermultiplet manifolds is to construct a new version of the so-called c-map, which is well known from minkowskian theories [25, 26, 14, 21]. In fact, we will construct *two* new c-maps, called the minkowskian and the euclidean para-c-map (for reasons that will become obvious in a moment). Each

[1] We refer to [12] for a review.

of the c-maps is constructed using two complementary approaches, a physical and a mathematical one. While the physical approach, based on T-duality and dimensional reduction of lagrangians, provides an explicit expression for the hypermultiplet metric, the geometrical structures of the hypermultiplet manifolds are not manifest and their identification requires a lot of work. The mathematical approach provides a geometrical construction of the hypermultiplet manifolds and the origin of all its structures is transparent and manifest. When formulated in terms of coordinates which correspond to the physical scalar fields, it is straightforward to verify that the metrics obtained in both constructions are identical.

For completeness, let us also mention that there is another construction which yields a formal power series for a hyper-Kähler metric along the zero section of the cotangent bundle of a Kähler manifold [27]. Under the assumption that the Kähler metric is real analytic, the formal power series converges to a true hyper-Kähler metric in some neighborhood of the zero section. Special Kähler metrics are real analytic, since a holomorphic prepotential defines a real analytic Kähler potential. Therefore, the above construction provides a hyper-Kähler metric in some neighborhood of the zero section of the cotangent bundle of any special Kähler manifold. On the other hand, the c-map defines a global hyper-Kähler metric and the question arises, whether the c-map extends the local hyper-Kähler metric of [27]. This is indeed the case, as the hyper-Kähler metric obtained by the c-map verifies the assumptions of the uniqueness result of [27]: it is U(1)-invariant and restricts to the given Kähler metric on the zero section. Using superspace methods, four-dimensional $N = 2$ supersymmetric sigma models, whose target spaces are the cotangent bundles of Kähler manifolds, have been constructed in [28]. This method gives an expression for the hyper-Kähler potential, which still depends on auxiliary fields. While for some special cases it is possible to eliminate the auxiliary fields and to obtain the metric explicitly, it appears that in general the elimination can only be performed through an expansion in an auxiliary variable.

The outline of this paper is as follows: in the reminder of this secion we give a comprehensive summary and discussion of the results, and we also give a brief outlook on future directions of research. Section 2 studies the c-maps from the mathematical point of view. Here we provide the definitions for all the relevant notions of para-complex and para-hypercomplex geometry, and we formulate and prove five theorems which specify the properties of the c-maps. Section 3 gives the physical treatment of the c-maps through the dimensional reduction of supersymmetric lagrangians from four to three dimensions. The resulting hypermultiplet metric is shown to be of the type constructed in section 2. In section 4 we discuss a restricted class of hypermultiplet manifolds which can be obtained by dimensional reduction from five to three dimensions. The reduction from five to four dimensions is related to the so-called r-maps, and we can summarize the relations between r-maps and c-maps in a commutative diagram.

1.2 Summary and discussion

The physical way to understand the c-map is its relation to T-duality in string theory. To be concrete let us consider type-IIA and type-IIB string theory, both in a space-time background of the form $M_d \times X_{10-d}$, where M_d is d-dimensional Minkowski space and X_{10-d} is a $(10 - d)$-dimensional Ricci-flat compact manifold. Then T-duality states that type-IIA string theory in the background $M_{d-1} \times S_1^R \times X_{10-d}$ is identical to type-IIB string theory in the background $M_{d-1} \times S_1^{R^{-1}} \times X_{10-d}$. Here M_{d-1} is $(d-1)$-dimensional Minkowski space, S_1^R is a circle of radius R, and $S_1^{R^{-1}}$ is a circle with the inverse radius. (The radius is measured in terms of the fundamental length scale $\sqrt{\alpha'}$ of string theory.) In other words the dimensional reduction from d to $d-1$ dimensions yields *one* continuous family of theories, labelled by R, which has two distinct d-dimensional limits $R \to \infty$ (IIA theory on $M_d \times X_{10-d}$) and $R^{-1} \to \infty$ (IIB theory on $M_d \times X_{10-d}$):

$$\mathrm{IIA}/X_{10-d} \leftarrow \mathrm{IIA}/(X_{10-d} \times S_1^R) = \mathrm{IIB}/(X_{10-d} \times S_1^{R^{-1}}) \to \mathrm{IIB}/X_{10-d}\,.$$

To obtain the c-map it is sufficient to go from the full string theories to the d-dimensional effective supergravity theories which describe their massless modes. We now choose $d = 4$ and take X to be a (generic) Calabi-Yau threefold X_6. Then one has two four-dimensional supergravity theories with $\mathcal{N} = 2$ supersymmetry (8 supercharges) which are related as follows: given one of the two supergravity actions, one performs a dimensional reduction on a circle of radius R and obtains a three-dimensional supergravity action. Then one performs the limit $R^{-1} \to \infty$ and re-interprets the result as a four-dimensional theory. In practise, this can be done by finding a four-dimensional theory which reproduces the three-dimensional theory upon dimensional reduction on a circle of radius R^{-1}. In the case at hand, this construction maps the vector multiplets (hypermultiplets) of the four-dimensional type-IIA theory to the hypermultiplets (vector multiplets) of the four-dimensional type-IIB theory. Recall that for Minkowski signature a four-dimensional vector multiplet (A_μ, λ_i, z) consists of a gauge field A_μ, a doublet of Majorana spinors λ_i, $i = 1, 2$ and a complex scalar z. Under dimensional reduction the vector A_μ decomposes into a scalar A_3 and a vector A_m, $m = 0, 1, 2$. Moreover, a three-dimensional vector field can be dualized into a scalar field,[2] so that all together we obtain a three-dimensional hypermultiplet which consists of four real scalars and a doublet of Majorana spinors. In the limit $R^{-1} \to \infty$ this becomes a four-dimensional hypermultiplet with the same scalar manifold. This defines a map, known as the c-map, between the vector multiplet manifold M of the four-dimensional IIA theory and the hypermultiplet manifold N of the four-dimensional IIB theory. Note that the c-map is already determined by the dimensional reduction of the bosonic part of the vector multiplet lagrangian from four to three dimensions:

$$c : M = \{\text{mfd. of 4d vector multiplet scalars}\} \to N = \{\text{mfd. of 3d/4d hypermultiplet scalars}\}\,.$$

The opposite happens for the hypermultiplets of the IIA theory. Under dimensional reduction they become three-dimensional hypermultiplets, but in order to perform the limit $R^{-1} \to \infty$ one needs to dualize one of the scalars into a three-dimensional vector, which becomes a four-dimensional vector in the decompactification limit. This way the hypermultiplets of the IIA theory are mapped to the vector multiplets of the IIB theory. The resulting map between the scalar manifolds is the inverse c-map.

So far we discussed the c-map in the context of supergravity, which is natural because string theory automatically incorporates gravity. But one can also consider a limit where

[2] See section 3 for the details.

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gravity decouples. Then one obtains a c-map, called the rigid c-map,[3] between the vector and hypermultiplet manifolds of rigidly supersymmetric theories. In four-dimensional rigid $\mathcal{N}=2$ superymmetry, the target space M of the vector multiplet scalars must be an affine special Kähler manifold of real dimension $2n$, where n is the number of vector multiplets. Since hypermultiplet manifolds must be hyper-Kähler, the c-map assigns to every affine special Kähler manifold M a hyper-Kähler manifold N of twice the dimension. Note, however, that not all hyper-Kähler manifolds can be obtained this way. In fact, the hyper-Kähler manifolds in the image of c-map are non-generic, because they have isometries originating from the gauge symmetries of the vector fields which have been dualized into scalars.

In this paper we will use two modified c-maps to construct euclidean hypermultiplet manifolds from vector multiplet manifolds. One way is to start from the vector multiplet lagrangians of euclidean four-dimensional theories, which were constructed in [1], and to reduce the theory to three dimensions. Again the four-dimensional vector decomposes into a three-dimensional vector and a scalar, and the vector can be dualized into another scalar. This way we obtain a three-dimensional euclidean hypermultiplet lagrangian, from which we can read off the hypermultiplet manifold. The resulting c-map is called the euclidean para-c-map, denoted c^{4+0}_{3+0}. While the initial vector multiplet manifold M is affine special para-Kähler, the resulting hypermultiplet manifold N is para-hyper-Kähler:

$$c^{4+0}_{3+0} : \{\text{affine special para-Kähler mfds.}\} \to \{\text{para-hyper-Kähler mfds.}\}\,. \tag{1.1}$$

The second option that we have to construct euclidean hypermultiplet manifolds is to start with a minkowskian vector multiplet lagrangian and to perform a dimensional reduction over time. This defines a map c^{3+1}_{3+0}, called the minkowskian para-c-map which assigns to every minkowskian vector multiplet manifold a euclidean hypermultiplet manifold. While the minkowskian vector multiplet manifolds are special Kähler manifolds, the euclidean hypermultiplet manifolds are again para-hyper-Kähler manifolds:

$$c^{3+1}_{3+0} : \{\text{affine special Kähler mfds.}\} \to \{\text{para-hyper-Kähler mfds.}\}\,. \tag{1.2}$$

However, para-hyper-Kähler manifolds which are constructed using c^{3+1}_{3+0} can generically not be obtained from c^{4+0}_{3+0}. As we will show in section 4, the manifolds which are in the image of both c-maps are precisely those which can be obtained by dimensional reduction of a five-dimensional (minkowskian) vector multiplet lagrangian with respect to one time-like and one space-like direction. This observation combines nicely with the results of [1], where we worked out and compared the dimensional reduction of vector multiplets from $4+1$ to $3+1$ dimensions (reduction over space) and to $4+0$ dimensions (reduction over time). Vector multiplets in $4+1$ dimensions contain one real scalar, and the scalar manifolds are so-called affine very special real manifolds. For such manifolds the metric is encoded in a real polynomial of degree 3. By dimensional reduction over space (time) the scalars become complex (para-complex) and the resulting scalar manifolds are affine special Kähler and affine special para-Kähler, respectively. The dimensional reduction defines the

[3] We will omit the 'rigid' in the following.

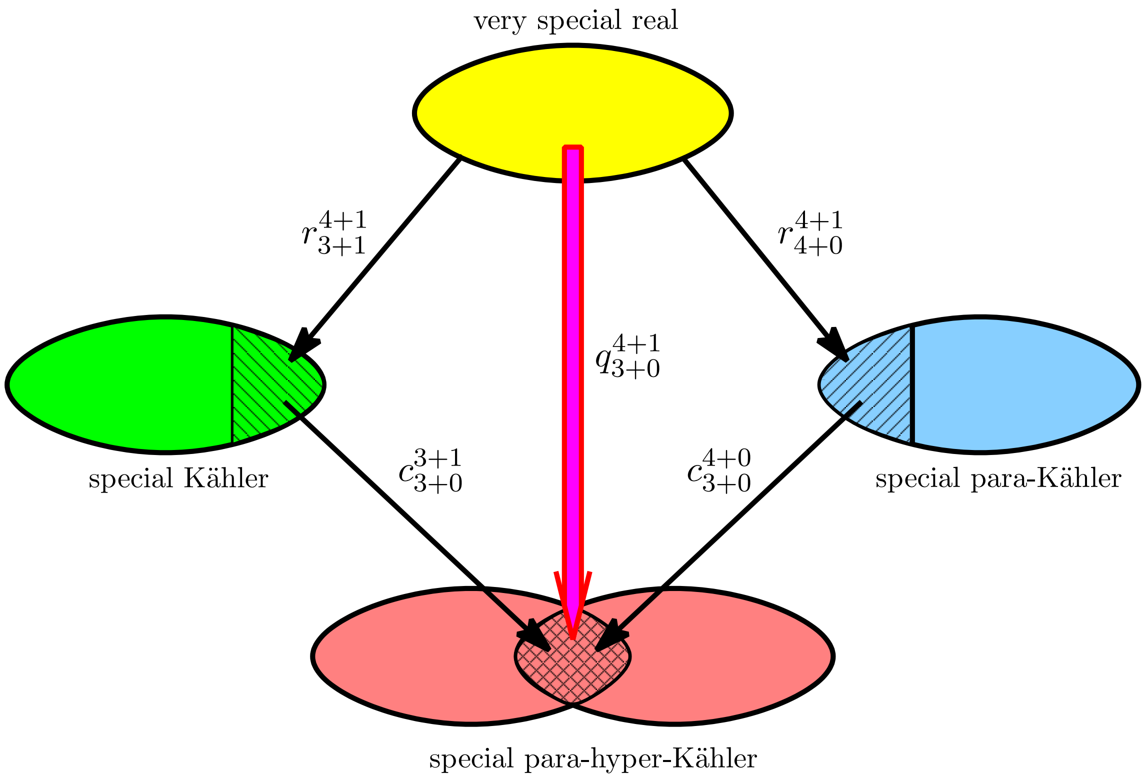

Figure 1: This figure summarizes how the scalar manifolds of three-dimensional euclidean hypermultiplets (at the bottom) are related to those of minkowskian and euclidean four-dimensional vector multiplets (in the middle) and of five-dimensional minkowskian vector multiplets (at the top). Dimensional reduction does only yield a special subset of the lower dimensional theories: the scalars of five-dimensional vector multiplets parametrize a very special real manifold, which is encoded in a real polynomial of degree 3, the prepotential. The reduction over space (time) gives very special (para-)Kähler manifolds (hatched segments of left and right middle blob, respectively). These are determined by the real cubic prepotential of the five-dimensional theory, while generic special (para-)Kähler manifolds have a general (para-)holomorphic prepotential (full blobs in the middle). The reduction of general four-dimensional minkowskian (euclidean) vector multiplets over time (space) gives special para-hyper-Kähler manifolds (intersecting blobs at the bottom), which are determined by the (para-)holomorphic prepotential of (at least) one of the four-dimensional theories. The reduction from five to three dimensions yields very special para-hyper-Kähler manifolds (doubly-hatched segment of the lower blobs), which are encoded in a real cubic prepotential.

so-called r-maps, r^{4+1}_{3+1} and r^{4+1}_{4+0} between the respective scalar manifolds. If one performs a reduction from $4+1$ to $3+0$ dimensions, the result is independent of whether one reduces first over space or time, and the corresponding para-hyper-Kähler manifolds are isometric. This defines a map, called the para-q-map,

$$q^{4+1}_{3+0} := c^{3+1}_{3+0} \circ r^{4+1}_{3+1} \cong c^{4+0}_{3+0} \circ r^{4+1}_{4+0}\,, \tag{1.3}$$

which assigns to each very special real manifold a para-hyper-Kähler manifold. The mutual relations between the various scalar manifolds are summarized in figure 1.

While the physical approach to the c-map provides explicit expressions for the metric of the hypermultiplet manifold (and, if we include the fermions, for the whole lagrangian) its geometrical structure is not manifest. If we take, for concreteness, the dimensional reduction of a minkowskian lagrangian with n vector multiplets, then we start with $2n$ real scalar fields. However, the manifold M spanned by these scalars is affine special Kähler and in particular complex. Thus we have a complex structure J, $J^2 = -\mathbb{1}$. After dimensional reduction over time and dualization of the vector fields, we have a scalar manifold N of real dimension $4n$. One can then show that the real scalars can be combined into $2n$ complex fields, i.e., the complex structure J of M induces a complex structure J_1 on N. But this already requires to identify the proper way of combining real fields into complex ones, see section 3 for the details. In order to prove that N is para-hyper-Kähler, we have to work even harder: first, we have to find a para-complex structure J_2, which anticommutes with J_1. This gives us a second para-complex structure $J_3 = J_1 J_2$ for free, and J_1, J_2, J_3 satisfy the para-quaternionic algebra. Second, we have to show that the metric is para-hyper-hermitean, i.e., J_2, J_3 must be anti-isometries and J_1 must be an isometry. Third, the metric is even para-hyper-Kähler, i.e., the structures J_1, J_2, J_3 must be covariantly constant with respect to the Levi-Civita connection. The main problem with this approach is that there is no systematic procedure to identify J_1 and J_2. A strategy used in the literature is to show that the metric has the appropriate restricted holonomy group [25, 26]. Here one uses that a riemannian manifold of dimension $4n$ is hyper-Kähler if and only if its holonomy group is contained in $\mathrm{USp}(2n)$ (the compact real form of $\mathrm{Sp}(\mathbb{C}^{2n})$). Analogously a riemannian manifold of dimension $4n$ is para-hyper-Kähler if and only if its holonomy group is contained in the real symplectic group $\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Sp}(\mathbb{R}^{2n}) = \mathrm{Id}_{\mathbb{R}^2} \otimes \mathrm{Sp}(\mathbb{R}^{2\mathrm{n}}) \subset \mathrm{GL}(\mathbb{R}^2 \otimes \mathbb{R}^{2\mathrm{n}}) = \mathrm{GL}(4\mathrm{n},\mathbb{R})$. However, the only efficient way to show that the holonomy group is contained in $\mathrm{USp}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$, respectively, is to show that the structures J_α are parallel. Thus the identification of these structures is indeed the main problem.

Here lies the great advantage of the geometrical construction which we present in section 2. The basic result is that if M is any special para-Kähler manifold (theorem 2) or any special Kähler manifold (theorem 3),[4] then its cotangent bundle $N = T^*M$ is a para-hyper-Kähler manifold. All the geometrical data of N, including the structures J_1, J_2, J_3 can be constructed in terms of the special geometry data of M. The proof of the central theorems 2 and 3 requires some preliminaries. In section 2.1 we review the relevant facts and definitions about para-complex and para-hypercomplex geometry, and about special (para-)Kähler manifolds. Sections 2.2 and 2.3 derive properties of the cotangent bundle $N = T^*M$ of a para-complex manifold M. In particular, we show that N has a

[4]This theorem also applies to the case of indefinite signature (special pseudo-Kähler manifolds). These manifolds are not admissible target spaces for rigid vector multiplets, but play an important role when constructing the projective special Kähler manifolds which are the target spaces of locally supersymmetric vector multiplets [18]. This is well known in the context of the superconformal tensor calculus, where the metric of vector multiplet scalars is indefinite before fixing the gauge for dilatations and U(1) transformations [29]. The indefiniteness of the metric reflects that some of the fields act as superconformal compensators. We expect that pseudo-Kähler manifolds will appear, when constructing the local version of the para-c-map. This will be addressed in a future publication.

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canonical para-complex structure J_N. If M is equipped with a linear connection ∇, one can decompose the tangent bundle TN of N into a horizontal and vertical part which are isomorphic to the pullbacks of the tangent bundle TM and of the cotangent bundle T^*M of M, respectively. Moreover ∇ can be used to define another para-complex structure J^∇ on N. We derive a sufficent condition for ∇, which implies that J^∇ is the canonical para-complex structure J_N. It turns out that this condition is met if M is special para-Kähler and ∇ is its special connection. In section 2.4 we introduce a metric on M. If M is (almost) para-hermitean, then one can find a metric g_N on N and a second para-complex structure J^ω on N, such that N is an (almost) para-hyper-hermitean manifold. This is the subject of theorem 1. In section 2.5 we take M to be a special para-Kähler manifold and prove theorem 2, which states that N is para-hyper-Kähler. We also formulate theorem 3, which claims the same result if M is special (pseudo-)Kähler. The proof is omitted because it is completely analogous to the one of theorem 2. At this point we have already the complete geometrical picture. In sections 2.6 and 2.7 we reformulate the constructions behind theorems 2 and 3 in terms of (para-)holomorphic coordinates. The action of the structures J_α in the (para-)holomorphic basis is given in theorems 4 and 5, respectively. We also work out the explicit expressions for the metric g_N in terms of (para-)holomorphic coordinates. In this parametrization it is easy to compare our results with the Minkowski case [25].

1.3 Outlook

Since we only discuss the c-map for rigidly supersymmetric theories in this paper, the natural next step is to consider hypermultiplets in supergravity. It will be interesting to have a fresh look on the minkowskian version of the local c-map, because the geometrical methods used and developed in this paper apply to this case as well. We expect that the geometrical structures behind the c-map can be made more transparent. Here the recent work on superconformal hypermultiplets [30] should be helpful. Given that not much is known about higher-dimensional hypermultiplet manifolds in (minkowskian) supergravity, this is worthwhile to investigate. Moreover we can treat the minkowskian and the euclidean case in parallel.

Another extension of the results of this paper is to include the fermionic degrees of freedom. This is necessary in view of our ultimate goal, the construction of supersymmetric instanton solutions. We also remark that so far we only constructed a particular subset of the possible euclidean hypermultiplet manifolds, namely those in the image of the para-c-map. But by analogy to minkowskian hypermultiplets one expects that any para-hyper-Kähler manifold defines a supersymmetric hypermultiplet lagrangian. The corresponding statement for vector multiplets was proven in [1]: any special para-Kähler manifold, not just those in the image of the r-map (i.e., those obtainable by dimensional reduction from five dimensions) defines a euclidean vector multiplet lagrangian, and, moreover, these are the most general admissible scalar manifolds. The idea of the proof was to rewrite the euclidean lagrangian and supersymmetry transformation rules in such a way that they took the same form as their minkowskian counterparts. We expect that this approach applies to hypermultiplets as well.

2. Geometrical construction of the para-C-maps

2.1 Definitions and basic facts

Let us first recall some definitions and basic facts about para-complex and para-hyper-complex geometry, see [1] for more on (special) para-Kähler manifolds and [31] for (symmetric) para-hyper-Kähler manifolds.

Definition 1. *An* almost para-complex structure *on a smooth manifold M is an endomorphism field $J \in \Gamma(\operatorname{End} TM)$, such that*

(i) $J \neq \mathrm{Id}_{TM}$ is an involution, i.e., $J^2 = \mathrm{Id}_{TM}$ and

(ii) the two eigendistributions $T^{\pm}M := \ker(\mathrm{Id} \mp J)$ of J have the same rank.

An almost para-complex structure J is called integrable *if the distributions $T^{\pm}M$ are both integrable. A* para-complex structure *is an integrable almost para-complex structure. A manifold M endowed with an (almost) para-complex structure J is called and* (almost) para-complex manifold*. A map $f : (M, J) \to (M', J')$ between (almost) para-complex manifolds is called* para-holomorphic *if $df\, J = J' df$.*

Notice that the endomorphism $J_p \in \operatorname{End} T_pM$, $p \in M$, defines on the tangent space T_pM the structure of a free module of rank $n = \dim T_p^{\pm}M$ over the ring of para-complex numbers

$$C := \mathbb{R}[e] = \{a + eb \,|\, a, b \in \mathbb{R}\}\,, \quad e^2 = 1\,. \tag{2.1}$$

The free module C^k, $k \in \mathbb{N}$, is itself an example of a para-complex manifold, the para-complex structure being the multiplication by e. In particular, we can speak of *para-holomorphic functions $f : (M, J) \to C$.*

Every para-complex manifold (M, J) admits a system of so-called *adapted local coordinates* $(z_+^1, \ldots, z_+^n, z_-^1, \ldots, z_-^n)$, such that the z_+^i (respectively, z_-^i) are constant on the leaves of the integrable distribution T^-M (respectively, T^+M). One can check that the C-valued functions

$$z^i := \frac{z_+^i + z_-^i}{2} + e\frac{z_+^i - z_-^i}{2} \tag{2.2}$$

are para-holomorphic. They form what is called a system of *para-holomorphic local coordinates.*

Definition 2. *A pseudo-riemannian metric g on an almost para-complex manifold (M, J) is called* para-hermitean *if J is skew-symmetric with respect to g. An* (almost) para-hermitean manifold *is an (almost) para-complex manifold (M, J) endowed with a para-hermitean metric g. The* fundamental two-form *of an almost para-hermitean manifold (M, J, g) is the non-degenerate skew-symmetric bilinear form $\omega = g(J\cdot, \cdot)$. A* para-Kähler manifold *is a para-hermitean manifold (M, J, g) for which the fundamental two-form ω is closed. In that case, the symplectic form ω is called the* para-Kähler form *of (M, J, g).*

Notice that the skew-symmetry of J with respect to g implies that J is an *anti-isometry*:

$$J^*g = g(J\cdot, J\cdot) = -g\,. \tag{2.3}$$

As for usual almost hermitean manifolds, the integrability of J together with the para-Kähler condition $d\omega = 0$ on an almost para-hermitean manifold (M, J, g) is equivalent to J being parallel for the Levi-Civita connection of (M, g).

Definition 3. *A* special para-Kähler manifold[5] (M, J, g, ∇) *is a para-Kähler manifold* (M, J, g) *endowed with a flat torsion-free connection* ∇ *such that*

(i) ∇ *is symplectic with respect to the para-Kähler form, i.e.,* $\nabla\omega = 0$ *and*

(ii) ∇J *is a symmetric (1,2)-tensor field, i.e.,* $(\nabla_X J)Y = (\nabla_Y J)X$ *for all* X, Y.

It is proven in [1] that any simply connected special para-Kähler manifold (M, J, g, ∇) admits a canonical realization as a para-holomorphic lagrangian immersion $\phi : M \to V = C^{2n}$, which induces the special geometric structures on M. Here V is endowed with the standard para-holomorphic symplectic form Ω and the standard real structure, for which $\mathbb{R}^{2n} \subset C^{2n}$ is the subset of real points.

For any point $p \in M$, there exists a global system of linear para-holomorphic coordinates (z^i, w_i) on V which is compatible with the structures on V and has the property that the image $\phi(U)$ of some neighborhood $U \subset M$ of p is defined by a system of equations of the form

$$w_i = F_i := \frac{\partial F}{\partial z^i}\,, \tag{2.4}$$

where $F = F(z^1, \ldots, z^n)$ is a (locally defined) para-holomorphic function of n variables. F is called the para-holomorphic *prepotential*. The compatibility of (z^i, w_i) with the structures on V means that the coordinates are *canonical* for Ω, i.e., $\Omega = \sum dz^i \wedge dw_i$, and real-valued on $\mathbb{R}^{2n} \subset C^{2n} = V$.

Next we provide the basic definitions of para-hypercomplex geometry.

Definition 4. *An* (almost) para-hypercomplex manifold *is a smooth manifold M endowed with an* (almost) para-hypercomplex structure*, i.e., with three pairwise anticommuting endomorphism fields* $J_1, J_2, J_3 = J_1 J_2 \in \Gamma(\mathrm{End}\, TM)$*, such that two of them are (almost) para-complex structures and one of them is an (almost) complex structure.*

An (almost) para-hyper-hermitean manifold *is an (almost) para-hypercomplex manifold* (M, J_1, J_2, J_3) *endowed with a pseudo-riemannian metric g for which the three endomorphism fields J_α are skew-symmetric. The quadruplet* (J_1, J_2, J_3, g) *is called an* (almost) para-hyper-hermitean structure.

A para-hyper-hermitean manifold or structure is called para-hyper-Kähler *if the three fundamental two-forms* $\omega_\alpha = g(J_\alpha\cdot, \cdot)$ *are closed.*

[5] More precisely, an *affine* special para-Kähler manifold.

2.2 The cotangent bundle of a para-complex manifold

Let (M, J) be a para-complex manifold. Its cotangent bundle $N = T^*M$ (as well as its tangent bundle) carries a canonical para-complex structure J_N such that the canonical projection $\pi : N \to M$ is para-holomorphic, i.e., $d\pi J_N = J d\pi$. Adapted local coordinates for J_N can be constructed as follows. Let $(z^i_\pm)_{i=1,\ldots,n}$ be a system of adapted local coordinates for (M, J) defined on an open set $U \subset M$. We can consider them as functions on the open set $\pi^{-1}(U) \subset N$ via the projection $\pi : N \to M$. We define new functions $w^\pm_i$ on $\pi^{-1}(U)$ which are linear on the fibers of π and satisfy

$$w^\pm_i(dz^j_\pm) = \delta^j_i\,, \qquad w^\pm_i(dz^j_\mp) = 0\,. \tag{2.5}$$

Then $(z^i_\pm, w^\pm_i)_{i=1,\ldots,n}$ is a system of adapted local coordinates on (N, J_N), i.e.,

$$J_N\left(\frac{\partial}{\partial z^i_\pm}\right) = \pm\frac{\partial}{\partial z^i_\pm}\,, \qquad J_N\left(\frac{\partial}{\partial w^\pm_i}\right) = \pm\frac{\partial}{\partial w^\pm_i}\,. \tag{2.6}$$

2.3 The almost para-complex structure on N associated to a connection on (M, J)

A linear connection ∇ on M defines a decomposition

$$T_\xi N = \mathcal{H}^\nabla_\xi \oplus T^v_\xi N \cong T_pM \oplus T^*_pM\,, \qquad \xi \in N\,, \qquad p = \pi(\xi)\,, \tag{2.7}$$

where $\pi : N = T^*M \to M$ is the canonical projection, the vertical space

$$T^v_\xi N := \ker d\pi_\xi = T_\xi N_p \cong N_p \tag{2.8}$$

is canonically identified with the vector space $N_p = T^*_pM$ and the horizontal space $\mathcal{H}^\nabla_\xi$ is identified with the tangent space to the basis via the isomorphism

$$d\pi_\xi|_{\mathcal{H}^\nabla_\xi} : \mathcal{H}^\nabla_\xi \xrightarrow{\sim} T_pM\,. \tag{2.9}$$

Notice that the vertical subbundle $T^vN \subset TN$ is J_N-invariant, since the projection π is para-holomorphic, whereas the horizontal subbundle $\mathcal{H}^\nabla \subset TN$ is in general not J_N-invariant. This allows us to define a second *almost* para-complex structure J^∇ by

$$J^\nabla_\xi = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}\,, \tag{2.10}$$

with respect to the canonical identification $T_\xi N \cong T_pM \oplus T^*_pM$, explained above. Here we used the standard notation $J^*\alpha = \alpha \circ J$, for any $\alpha \in T^*_pM$. More generally, we can define J^∇ for any *almost* para-complex structure J. However, in the following we will assume that J is integrable. J^∇ coincides with the canonical para-complex structure J_N only if $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla$. In this section we discuss necessary and sufficient conditions for the horizontal distribution $\mathcal{H}^\nabla$ to be J_N-invariant.

Definition 5. *A linear connection D on a para-complex manifold (M, J) is called a* para-complex connection *if it is torsion-free and satisfies $DJ = 0$.*

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Proposition 1. *Let ∇ be a linear connection on a para-complex manifold (M, J). Then the horizontal distribution $\mathcal{H}^\nabla$ is invariant under the canonical para-complex structure J_N on $N = T^*M$ if and only if there exists a para-complex connection D on (M, J) such that $A := \nabla - D$ satisfies*

$$A_X \circ J = A_{JX}\,, \tag{2.11}$$

for all $X \in TM$.

Proof. We will use the following standard lemma which relates the horizontal distributions with respect to two connections on the same manifold.

Lemma 1. *Let ∇, D be two linear connections on a manifold M and $A := \nabla - D \in \Omega^1(\mathrm{End}\,TM)$ their difference tensor. Then the corresponding horizontal distributions $\mathcal{H}^\nabla$, $\mathcal{H}^D \subset TN$ in the cotangent bundle $N = T^*M$ are related by*

$$\mathcal{H}^\nabla_\xi = \{\hat{v} := v + A^\xi_v \mid v \in \mathcal{H}^D_\xi \cong T_pM\}\,, \qquad \xi \in N\,, \quad p = \pi(\xi)\,, \tag{2.12}$$

where $A^\xi_v := A^*_v\xi = \xi \circ A_v \in T^*_pM \cong T^v_\xi N$.

As a first step in the proof of proposition 1, let us first settle the case when ∇ is a para-complex connection.

Proposition 2. *Let ∇ be a para-complex connection on a para-complex manifold (M, J). Then $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla$ and, hence, $J_N = J^\nabla$.*

Proof. Any choice of adapted coordinates $(z^i_\pm)$ on an open set $U \subset M$ induces a flat para-complex connection D in the vector bundle $T^*M|_U = T^*U$, such that $Ddz^i_\pm = 0$. With respect to the induced adapted coordinates $(z^i_\pm, w^\pm_i)$ on $\pi^{-1}(U) \subset N$, the horizontal and vertical distributions are simply

$$\mathcal{H}^D = \mathrm{span}\{\frac{\partial}{\partial z^i_\pm} \mid i = 1, \ldots, n\} \qquad \text{and} \qquad T^vN = \mathrm{span}\{\frac{\partial}{\partial w^\pm_i} \mid i = 1, \ldots, n\}\,. \tag{2.13}$$

In particular, $J_N\mathcal{H}^D = \mathcal{H}^D$, and $J_N = J^D$, cf. (2.6). The tensor $A = \nabla - D$ satisfies

$$A_XY = A_YX \tag{2.14}$$

$$[A_X, J] = 0\,, \tag{2.15}$$

for all tangent vectors $X, Y \in T_pM$, $p \in M$, because ∇ and D are para-complex. The last equation can be reformulated as

$$J^*A^\xi_X = A^{J^*\xi}_X\,, \tag{2.16}$$

for all $\xi \in T^*M$, $X \in T_{\pi(\xi)}M$.

Now we check that (2.14) and (2.16) imply the J_N-invariance of $\mathcal{H}^\nabla$. For any $\hat{v} = v + A^\xi_v \in \mathcal{H}^\nabla_\xi$, $v \in \mathcal{H}^D_\xi \cong T_{\pi(\xi)}M$, see Lemma 1, we calculate

$$\begin{aligned} J_N\hat{v} &= J^D(v + A^\xi_v) = Jv + J^*A^\xi_v \overset{(2.16)}{=} Jv + A^{J^*\xi}_v \overset{(2.14)}{=} Jv + A^{J^*\xi}v \\ &\overset{(2.16)}{=} Jv + A^\xi(Jv) \overset{(2.14)}{=} Jv + A^\xi_{Jv} = \widehat{Jv} \in \mathcal{H}^\nabla_\xi\,. \end{aligned}$$

This proves proposition 2. □

Now we prove proposition 1. Let ∇ be a connection on (M, J) such that $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla$. The integrability of J implies the existence of a para-complex connection D on (M, J). In fact, as explained above, any adapted local coordinate system on (M, J) defines locally a flat para-complex connection. Pasting these locally defined connections by a smooth partition of unity, we obtain a globally defined para-complex connection D (which, in general, is not flat). By proposition 2, we have $J_N = J^D$, which allows us to compute for $\hat{v} = v + A^\xi_v \in \mathcal{H}^\nabla_\xi$, ($v \in \mathcal{H}^D_\xi$, $\xi \in N$):

$$J_N\hat{v} = J^D(v + A^\xi_v) = Jv + J^*A^\xi_v\,.$$

Since $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla = \{\hat{v} \mid v \in \mathcal{H}^D\}$, we conclude that

$$J^*A^\xi_v = A^\xi_{Jv}\,.$$

This proves that A satisfies (2.11).

Conversely, let D be a para-complex connection on (M, J) and ∇ any linear connection on M such that $A = \nabla - D$ satisfies (2.11). Again, by proposition 2, we have $J_N = J^D$, and

$$J_N\hat{v} = J^D(v + A^\xi_v) = Jv + J^*A^\xi_v \overset{(2.11)}{=} Jv + A^\xi_{Jv} = \widehat{Jv}\,,$$

for all $v \in \mathcal{H}^D_\xi$, $\xi \in N$. This shows that $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla$. □

Proposition 3. *Let (M, J) be a para-complex manifold and ∇ a torsion-free connection on M such that ∇J is symmetric. Then $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla$ and, hence, $J_N = J^\nabla$.*

Proof. The connection

$$\nabla^{(J)} := J \circ \nabla \circ J^{-1} = J \circ \nabla \circ J = \nabla + J(\nabla J) \tag{2.17}$$

is torsion-free, since ∇ is torsion-free and ∇J is symmetric. Therefore

$$D := \frac{1}{2}(\nabla + \nabla^{(J)}) = \nabla + \frac{1}{2}J(\nabla J) \tag{2.18}$$

is a torsion-free connection. We claim that D is para-complex. In fact, for all $X \in TM$, we have

$$\begin{aligned} D_XJ &= \nabla_XJ + \frac{1}{2}[J(\nabla_XJ), J] \\ &= \nabla_XJ + \frac{1}{2}(J(\nabla_XJ)J - \nabla_XJ) \\ &= \nabla_XJ - \nabla_XJ = 0\,. \end{aligned}$$

Here we used that $J^2 = \mathrm{Id}$ and, thus, $0 = \nabla_X J^2 = (\nabla_X J)J + J\nabla_X J$. Now, in view of proposition 1, it suffices to show that

$$A = \nabla - D = -\frac{1}{2}J(\nabla J) \tag{2.19}$$

satisfies (2.11), which is equivalent to

$$(\nabla_X J) \circ J = \nabla_{JX} J\,, \qquad \text{for all} \qquad X \in TM\,. \tag{2.20}$$

This follows from the symmetry of ∇J:

$$(\nabla_X J) \circ J = -J \circ (\nabla_X J) = -J(\nabla J)X = (\nabla J)JX = \nabla_{JX} J\,.$$

□

2.4 The almost para-hyper-hermitean structure on N associated to a connection and a para-hermitean metric on (M, J)

Let (M, J, g) be an almost para-hermitean manifold and ∇ a linear connection on M. Then we have the almost para-complex structure J^∇ on $N = T^*M$, see (2.10). Using the fundamental two-form $\omega = g(J\cdot, \cdot)$, we define a second almost para-complex structure J^ω on N by

$$J^\omega_\xi = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{2.21}$$

with respect to the canonical identification $T_\xi N \cong T_pM \oplus T_p^*M$ (2.7). We define also a pseudo-riemannian metric g_N on N by

$$g_N = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}. \tag{2.22}$$

Theorem 1. *Let (M, J, g) be an almost para-hermitean manifold endowed with a linear connection ∇. Then $(N, J_1 := J^\nabla, J_2 := J^\omega, J_3 := J_1 J_2, g_N)$ is an almost para-hyper-hermitean manifold.*

Proof. The skew-symmetry of J with respect to ω can be written as

$$J^* \circ \omega = -\omega \circ J\,, \tag{2.23}$$

if ω is considered as a map

$$\begin{aligned} \omega : TM &\to T^*M \\ v &\mapsto \omega(v, \cdot)\,. \end{aligned} \tag{2.24}$$

As a consequence, we obtain $\omega^{-1} \circ J^* = -J \circ \omega^{-1}$ and, hence, the identity $J_1 J_2 = -J_2 J_1$. This proves that (J_1, J_2, J_3) is an almost para-hypercomplex structure. It is clear that J_1 is skew-symmetric with respect to g_N, since J is g-skew-symmetric. It remains to check that J_2 is g_N-skew-symmetric. We have to check that the bilinear form

$$g_N \circ J_2 = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g \circ \omega^{-1} \\ g^{-1} \circ \omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g \circ \omega^{-1} \\ J & 0 \end{pmatrix} \tag{2.25}$$

is skew-symmetric. In other words, we have to check that

$$(g \circ \omega^{-1})^* = -J\,. \tag{2.26}$$

Notice that the symmetry of g and skew-symmetry of ω, can be expressed as

$$g^* = g \qquad \text{and} \qquad \omega^* = -\omega : TM = (T^*M)^* \to T^*M\,. \tag{2.27}$$

This yields the desired identity

$$(g \circ \omega^{-1})^* = (\omega^{-1})^* \circ g^* = -\omega^{-1} \circ g = -(g^{-1} \circ \omega)^{-1} = -J^{-1} = -J\,.$$

□

2.5 The para-hyper-Kähler structure on the cotangent bundle of a special para-Kähler manifold

Theorem 2. *Let (M, J, g, ∇) be a special para-Kähler manifold with para-Kähler form ω and (J_1, J_2, J_3, g_N) the almost para-hyper-hermitean structure on $N = T^*M$ constructed in theorem 1. Then (N, J_1, J_2, J_3, g_N) is a para-hyper-Kähler manifold.*

Proof. We have to show that the two almost para-complex structures J_1, J_2 and the almost complex structure J_3 are integrable and that the two-forms $\omega_\alpha := g_N(J_\alpha \cdot, \cdot)$, $\alpha = 1, 2, 3$, are closed. By proposition 3, $J_1 = J^\nabla$ coincides with the canonical (integrable) para-complex structure J_N. Since the symplectic form ω is ∇-parallel, the almost para-complex structure $J_2 = J^\omega$ is represented by a constant matrix with respect to any local ∇-affine coordinate system. This proves the integrability of J_1 and J_2. Let us now check that the corresponding fundamental two-forms are closed. With respect to the canonical identification $TN = \mathcal{H}^\nabla \oplus T^vN \cong TM \oplus T^*M$, we have

$$\omega_1 = g_N J_1 = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} = \begin{pmatrix} g \circ J & 0 \\ 0 & g^{-1} \circ J^* \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & -\omega^{-1} \end{pmatrix}, \tag{2.28}$$

where we have used the identity (2.26), which implies $g^{-1} \circ J^* = g^{-1} \circ (-g \circ \omega^{-1}) = -\omega^{-1}$. The equation (2.28) shows that ω_1 has constant coefficients with respect to any local ∇-affine coordinate system and is, therefore, closed. According to (2.25), we have that

$$\omega_2 = \begin{pmatrix} 0 & -J^* \\ J & 0 \end{pmatrix}. \tag{2.29}$$

To see that ω_2 is closed, let us choose ∇-affine local coordinates $(q^i)_{i=1,\dots,2n}$ on M and express J in these coordinates:

$$J\left(\frac{\partial}{\partial q^i}\right) = \sum J_i^j \frac{\partial}{\partial q^j}\,. \tag{2.30}$$

The expression for J^* in the dual co-frame (dq^i) is the transposed matrix of (J^i_j):

$$J^* dq^i = \sum J^i_j dq^j \,. \tag{2.31}$$

The canonical isomorphism $T^v_\xi N \cong T^*_p M$, $\xi \in N$, $p = \pi(\xi)$, maps

$$\left.\frac{\partial}{\partial p_i}\right|_\xi \mapsto dq^i|_p \,, \tag{2.32}$$

where $(q^i = q^i_N = \pi^* q^i_M, p_i)$ is the canonical system of local coordinates of $N = T^*M$ associated to the local coordinates $(q^i = q^i_M)$. Together with the equations (2.29)–(2.31) this shows that

$$\omega_2 = \sum J^i_j dq^j_N \wedge dp_i = \sum \pi^* \left(J^* dq^i_M\right) \wedge dp_i \,. \tag{2.33}$$

From $\nabla dq^i = 0$ and the symmetry of ∇J, we obtain that the one-forms $J^* dq^i$ on M are closed:

$$dJ^* dq^i = \mathrm{alt}\nabla(dq^i \circ J) = (dq^i) \circ \mathrm{alt}\nabla J = 0 \,. \tag{2.34}$$

Here we used the fact that the exterior derivative is the alternation (anti-symmetrization, up to a factor depending on the conventional identification between totally skew-symmetric tensors and exterior forms) of the covariant derivative for any torsion-free connection. Since the pull back of any closed form is closed, we obtain that the one-forms $\pi^*(J^* dq^i)$ on N are closed. This proves that ω_2 is closed. So we have proven that (N, J_1, g_N) and (N, J_2, g_N) are para-Kähler manifolds. In particular, J_1 and J_2 are parallel with respect to the Levi-Civita connection of (N, g_N). Therefore, $J_3 = J_1 J_2$ is also parallel, and (N, J_3, g_N) is an indefinite Kähler manifold. □

Similarly, we can prove the following theorem.

Theorem 3. *Let (M, J, g, ∇) be a special (pseudo-)Kähler manifold with Kähler form ω. Then the cotangent bundle $N = T^*M$ carries a para-hyper-Kähler structure (J_1, J_2, J_3, g_N), associated to the special Kähler structure on M. With respect to the canonical identification (2.7), the complex structure J_1 is given by*

$$J_1 := J^\nabla = J_N = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} , \tag{2.35}$$

the two para-complex structures J_2, J_3 are

$$J_2 := J^\omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} , \qquad J_3 := J_1 J_2 \tag{2.36}$$

and the para-hyper-Kähler metric is

$$g_N := \begin{pmatrix} g & 0 \\ 0 & -g^{-1} \end{pmatrix} . \tag{2.37}$$

Summarizing, we have defined two maps

$$c = c^{4+0}_{3+0} : \{\text{special para-Kähler manifolds}\} \to \{\text{para-hyper-Kähler manifolds}\} \quad (2.38)$$
$$c = c^{3+1}_{3+0} : \{\text{special Kähler manifolds}\} \to \{\text{para-hyper-Kähler manifolds}\} \quad (2.39)$$
$$(M, J, g, \nabla) \mapsto c(M, J, g, \nabla) := (N, J_1, J_2, J_3, g_N)\,, \quad (2.40)$$

which we call the (affine) *para-c-maps.* We can compose them with the two r-maps

$$r^{4+1}_{3+1} : \{\text{affine very special real manifolds}\} \to \{\text{special Kähler manifolds}\} \quad (2.41)$$
$$r^{4+1}_{4+0} : \{\text{affine very special real manifolds}\} \to \{\text{special para-Kähler manifolds}\} \quad (2.42)$$

(see [1] and references therein), obtaining two maps

$$q^{4+1}_{3+0} := c^{3+1}_{3+0} \circ r^{4+1}_{3+1} \quad \text{and} \quad c^{4+0}_{3+0} \circ r^{4+1}_{4+0} :$$
$$\{\text{affine very special real manifolds}\} \to \{\text{para-hyper-Kähler manifolds}\}\,. \quad (2.43)$$

We call q^{4+1}_{3+0} *the para-q-map.* This is justified by the following proposition, which follows from the discussion in section 4.

Proposition 4. *For any affine very special real manifold L, the para-hyper-Kähler manifolds $(c^{3+1}_{3+0} \circ r^{4+1}_{3+1})(L)$ and $(c^{4+0}_{3+0} \circ r^{4+1}_{4+0})(L)$ are canonically isometric.*

Remark 1. *Let (M, J, g, ∇) be a special (para-)Kähler manifold. Then*

$$(M, J, cg, \nabla^{(a,b)})\,, \quad \nabla^{(a,b)} := (a\mathrm{Id} + bJ) \circ \nabla \circ (a\mathrm{Id} + bJ)^{-1}\,, \quad (2.44)$$

is again a special (para-)Kähler manifold, for all $a, b, c \in \mathbb{R}$ such that

$$(a\mathrm{Id} + bJ)(a\mathrm{Id} - bJ) = \pm\mathrm{Id} \qquad \text{and} \qquad c \neq 0\,. \quad (2.45)$$

As a consequence, applying the para-c-map to the (non-connected) family $(M, J, cg, \nabla^{(a,b)})$ provides a family of para-hyper-Kähler structures on $N = T^*M$, which depends on two parameters. In addition, we also have the trivial freedom of multiplying the metric g_N by a non-zero constant.

Comparing the formulae (2.22) and (2.37) for the para-hyper-Kähler metrics g_N obtained by the para-c-maps c^{4+0}_{3+0} and c^{3+1}_{3+0}, respectively, one may wonder about the different sign in front of the inverse metric. As we shall explain now, the plus sign in front of g^{-1} in (2.22) can be converted into minus by a J_1-holomorphic diffeomorphism ψ of N. In fact, the map $J^* : T^*M \to T^*M$ defines a diffeomorphism ψ of $N = T^*M$. Its differential $d\psi$ preserves the vertical distribution T^vN and maps the horizontal distribution $\mathcal{H}^\nabla$ with respect to the connection ∇ to the horizontal distribution $\mathcal{H}^{\nabla'}$ with respect to the connection $\nabla' = J \circ \nabla \circ J^{-1}$. The differential $d\psi_\xi : T_\xi N \to T_\xi N$ at the point $\xi \in N$ is given by

$$d\psi_\xi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & J^* \end{pmatrix}, \quad (2.46)$$

where the domain and target are identified as (2.7) and

$$T_\xi N = \mathcal{H}_\xi^{\nabla'} \oplus T_\xi^v N \cong T_p M \oplus T_p^* M \,, \tag{2.47}$$

respectively. Using this, one can easily check that the diffeomorphism ψ transforms the para-hyper-Kähler structure (g_N, J_1, J_2, J_3) defined by the para-c-map c_{3+0}^{3+1}, with respect to (2.7), trivially to $(g_N, J_1, J_3, -J_2)$, with respect to (2.47), and maps the para-hyper-Kähler structure $(g_N = \mathrm{diag}(g, g^{-1}), J_1, J_2, J_3)$ obtained by the para-c-map c_{3+0}^{4+0} to a new para-hyper-Kähler structure $(g'_N = \mathrm{diag}(g, -g^{-1}), J'_1 = J_1, J'_2, J'_3)$, where

$$J'_2 = -\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \quad \text{and} \quad J'_3 = J'_1 J'_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} , \tag{2.48}$$

with respect to (2.47).

2.6 The euclidean para-c-map in the para-holomorphic parametrization

Later on we shall compare the para-c-map c_{3+0}^{4+0} introduced in the previous section to the sigma models which occur in the dimensional reduction of the euclidean vector multiplet lagrangian from four to three dimensions, followed by dualization into a hypermultiplet lagrangian. We shall find that they coincide up to a simple redefinition using the freedom discussed in (2.6) remark 1. For this it is useful to express the geometric data in terms of canonical para-holomorphic coordinates, which correspond to the para-complex scalar fields obtained from the dimensional reduction. This is accomplished in this section. The analogous results in the minkowskian case are given in the next section.

Let (M, J, g, ∇) be a special para-Kähler manifold and (J_1, J_2, J_3, g_N) the para-hyper-Kähler structure on $N = T^*M$ constructed in theorem 2. The geometric structure of M is locally completely specified by the para-holomorphic prepotential $F = F(z)$ with respect to some system of special para-holomorphic coordinates $z = (z^i)$. Let (z^i, w_i) be the canonical J_1-para-holomorphic local coordinates of

$$N = T^*M \cong \wedge^{1,0} T^*M \subset T^*M \otimes C = \wedge^{1,0} T^*M \oplus \wedge^{0,1} T^*M \tag{2.49}$$

associated to the special coordinates (z^i) on M.

Proposition 5. *The expression for the para-complex structure J_2 on N in the canonical para-holomorphic coordinates (z^i, w_i) is the following:*

$$J_2^* dz^i = -2e \sum N^{ij} \left(d\bar{w}_j - e\bar{F}_{jkl} N^{km} (w_m - \bar{w}_m) d\bar{z}^l \right) , \tag{2.50}$$

where (N^{ij}) is the inverse of the matrix (N_{ij}), with matrix elements $N_{ij} = e(F_{ij} - \bar{F}_{ij})$.

Proof. Let us first express the para-complex structure J_2 in canonical coordinates (q^a, p_a) on $N = T^*M$ associated to a system of ∇-affine local coordinates (q^a) on M. We choose

$$(q^a, p_a) = (x^i, y_i, \hat{x}_i, \hat{y}^i) \,, \tag{2.51}$$

where

$$z^i = x^i + eu^i \qquad \text{and} \qquad F_i = y_i + ev_i\,. \tag{2.52}$$

Then the para-Kähler form $\omega = g(J\cdot,\cdot)$ of M takes the form

$$\omega = 2\sum dx^i \wedge dy_i \tag{2.53}$$

and, hence,

$$J_2^* dx^i = \frac{1}{2} d\hat{y}^i\,, \quad J_2^* dy_i = -\frac{1}{2} d\hat{x}_i\,. \tag{2.54}$$

This allows us to compute the left-hand side of (2.50):

$$\begin{aligned} J_2^* dz^i &= J_2^*(dx^i + edu^i) = \frac{1}{2} d\hat{y}^i + \frac{e}{2}\left(\frac{\partial u^i}{\partial x^j}\, d\hat{y}^j - \frac{\partial u^i}{\partial y_j}\, d\hat{x}_j\right) \\ &= -\frac{e}{2}\frac{\partial u^i}{\partial y_j} d\hat{x}_j + \frac{1}{2}\left(\delta^i_j + e\frac{\partial u^i}{\partial x^j}\right) d\hat{y}^j\,. \end{aligned} \tag{2.55}$$

Notice that, in order to lighten the calculations, we are using Einstein's summation convention. The imaginary part of the equation $ddF = 0$ yields the useful identity

$$\frac{\partial u^i}{\partial x^j} = -\frac{\partial v_j}{\partial y_i}\,, \tag{2.56}$$

which shows that

$$\delta^i_j + e\frac{\partial u^i}{\partial x^j} = \delta^i_j - e\frac{\partial v_j}{\partial y_i} = \frac{\partial}{\partial y_i}\bar{F}_j = \bar{F}_{jk}\frac{\partial \bar{z}^k}{\partial y_i} = -e\bar{F}_{jk}\frac{\partial u^k}{\partial y_i}\,. \tag{2.57}$$

Decomposing the last equation in real and imaginary parts yields

$$R_{jk}\frac{\partial u^k}{\partial y_i} = -2\frac{\partial u^i}{\partial x^j} \quad \text{and} \quad N_{jk}\frac{\partial u^k}{\partial y_i} = 2\delta^i_j\,, \tag{2.58}$$

where

$$R_{ij} := F_{ij} + \bar{F}_{ij}\,. \tag{2.59}$$

In particular, we have

$$\frac{\partial u^i}{\partial y_j} = 2N^{ij}\,. \tag{2.60}$$

Using (2.55), (2.57) and (2.60) we can rewrite

$$\begin{aligned} J_2^* dz^i &= -eN^{ij} d\hat{x}_j - \frac{e}{2}\bar{F}_{jk}\frac{\partial u^k}{\partial y_i} d\hat{y}^j \\ &= -eN^{ij}\left(d\hat{x}_j + \bar{F}_{jk} d\hat{y}^k\right). \end{aligned} \tag{2.61}$$

In order to check that this coincides with the right-hand side of (2.50), let us first rewrite w_i as a function of the real canonical coordinates $(q^a, p_a) = (x^i, y_i, \hat{x}_i, \hat{y}^i)$. The real canonical coordinates (q^a, p_a) are related to the para-holomorphic canonical coordinates (z^i, w_i) via the identification

$$T^*M \xrightarrow{\sim} \wedge^{1,0} T^*M\,, \qquad \alpha = p_a dq^a \mapsto \frac{1}{2}(\alpha + eJ^*\alpha) = w_i dz^i\,. \tag{2.62}$$

We have

$$\alpha = p_a dq^a = \hat{x}_i dx^i + \hat{y}^i dy_i = \frac{1}{2}\hat{x}_i dz^i + \frac{1}{2}\hat{y}^i F_{ij} dz^j + \text{c.c.}\,. \tag{2.63}$$

Similarly, using the equations

$$J^* dx^i = du^i\,, \quad J^* dy_i = dv_i\,, \tag{2.64}$$

one obtains

$$eJ^*\alpha = \frac{1}{2}\hat{x}_i dz^i + \frac{1}{2}\hat{y}^i F_{ij} dz^j - \text{c.c.}\,. \tag{2.65}$$

Thus

$$w_i dz^i = \frac{1}{2}(\alpha + eJ^*\alpha) = \frac{1}{2}(\hat{x}_i + \hat{y}^j F_{ij}) dz^i\,. \tag{2.66}$$

This shows that

$$w_i = w_i(x, y, \hat{x}, \hat{y}) = \frac{1}{2}(\hat{x}_i + \hat{y}^j F_{ij})\,, \tag{2.67}$$

where $F_{ij} = F_{ij}(z)$ and $z^i = z^i(x, y)$. In particular,

$$e(w_i - \bar{w}_i) = \frac{1}{2}\hat{y}^j N_{ji} \tag{2.68}$$

and, hence,

$$dw_i - eF_{ijk}N^{jl}(w_l - \bar{w}_l)dz^k = dw_i - \frac{1}{2}\hat{y}^j F_{ijk} dz^k = \frac{1}{2}(d\hat{x}_i + F_{ij} d\hat{y}^j)\,. \tag{2.69}$$

Therefore, the right-hand side of (2.50) is given by

$$-2eN^{ij}(d\bar{w}_j - e\bar{F}_{jkl}N^{km}(w_m - \bar{w}_m)d\bar{z}^l) = -eN^{ij}(d\hat{x}_j + \bar{F}_{jk} d\hat{y}^k)\,. \tag{2.70}$$

Comparing this with (2.61) yields (2.50). □

In order to compute the para-hyper-Kähler metric of N in the para-holomorphic coordinates (z^i, w_i), it is useful to introduce a para-unitary co-frame

$$e^i = \sum e^i_I dz^I\,, \tag{2.71}$$

with respect to the sesquilinear para-hermitean metric[6]

$$h := g + e\omega = -\sum N_{IJ} dz^I d\bar{z}^J \tag{2.72}$$

on (M, J), i.e.,

$$N_{IJ} = -\sum e^i_I \bar{e}^i_J\,. \tag{2.73}$$

From now on, we shall distinguish the holonomic from the para-unitary co-frame by capital and lower case indices, respectively. We also put

$$E_i := 2\sum e^I_i(dw_I - eF_{IJK}N^{JL}(w_L - \bar{w}_L)dz^K)\,, \qquad (e^I_i) = (e^i_I)^{-1}\,. \tag{2.74}$$

Notice that $\sum e^i \wedge E_i = 2\sum dz^I \wedge dw_I$.

Now we express the full para-hyper-Kähler structure on N in the co-frame (e^i, E_i) of $\wedge^{1,0}_{J_1} T^* N$.

[6]The minus sign is due to the conventions of ch. 2 of [1], see equation (2.2) there.

Theorem 4. *Let (M, J, g, ∇) be a special para-Kähler manifold and (J_1, J_2, J_3, g_N) the para-hyper-Kähler structure on $N = T^*M$ constructed in theorem 2. In the co-frame (e^i, E_i), the para-hyper-Kähler structure has the following expression:*

(i) The para-hypercomplex structure $(J_1, J_2, J_3 = J_1 J_2)$ is given by

$$J_1^* e^i = e e^i \,, \qquad J_1^* E_i = e E_i \tag{2.75}$$
$$J_2^* e^i = e \bar{E}_i \,, \qquad J_2^* E_i = -e \bar{e}^i \,. \tag{2.76}$$

(ii) The J_1-sesquilinear para-hermitean metric $h_N := g_N + e\omega_1$, $\omega_1 = g_N(J_1 \cdot, \cdot)$, is given by

$$h_N = \sum (e^i \bar{e}^i + E_i \bar{E}_i) \,. \tag{2.77}$$

Proof. (i) follows from proposition 5.
(ii) h_N is the unique J_1-sesquilinear form such that $\mathrm{Re}\, h_N = g_N$. Therefore it is sufficient to check that

$$g_N = \mathrm{Re} \sum (e^i \bar{e}^i + E_i \bar{E}_i) \,. \tag{2.78}$$

In order to check this, we calculate the right-hand side in the real coordinates (2.51). The first term is

$$g = \mathrm{Re} \sum e^i \bar{e}^i \tag{2.79}$$
$$\begin{aligned} &= \mathrm{Re}\,(-N_{IJ} dz^I d\bar{z}^J) \\ &= -\left(N_{IJ} - \frac{\partial u^K}{\partial x^I} N_{KL} \frac{\partial u^L}{\partial x^J} \right) dx^I dx^J + 2 \left(\frac{\partial u^K}{\partial x^I} N_{KL} \frac{\partial u^L}{\partial y^J} \right) dx^I dy_J + \\ &\quad + \left(\frac{\partial u^K}{\partial y^I} N_{KL} \frac{\partial u^L}{\partial y^J} \right) dy_I dy_J \,. \end{aligned} \tag{2.80}$$

The second term is

$$\mathrm{Re} \sum E_i \bar{E}_i = \tag{2.81}$$
$$\begin{aligned} &= -4 N^{IJ} \left(dw_I - e F_{IKL} N^{LM} (w_M - \overline{w}_M) dz^L \right) \left(d\overline{w}_J - e \overline{F}_{JNP} N^{NQ} (w_Q - \overline{w}_Q)\, d\overline{z}^P \right) \\ &= -N^{IJ} d\hat{x}_I d\hat{x}^J - N^{IK} R_{KJ} d\hat{x}_I d\hat{y}^J + \frac{1}{4} (N_{IJ} - R_{IK} N^{KL} R_{LJ}) d\hat{y}^I d\hat{y}^J \,, \end{aligned} \tag{2.82}$$

where we used (2.69). Observe that (2.80) and (2.82) do not contain terms which mix (dx^I, dy_J) with $(d\hat{x}_I, d\hat{y}^J)$. For the following manipulations it is convenient to use matrix notation. Defining

$$P = (P^I_J) = \left(\frac{\partial u^I}{\partial x^J} \right) , \qquad Q = (Q^{IJ}) = \left(\frac{\partial u^I}{\partial y_J} \right) . \tag{2.83}$$

we can write

$$\mathrm{Re} \sum (e^i \bar{e}^i + E_i \bar{E}_i) = \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} , \tag{2.84}$$

where

$$g = \begin{pmatrix} -N + P^T N P & P^T N Q \\ Q^T N P & Q^T N Q \end{pmatrix} \tag{2.85}$$

and

$$g' = -\begin{pmatrix} N^{-1} & \frac{1}{2}N^{-1}R \\ \frac{1}{2}RN^{-1} & -\frac{1}{4}\left(N - RN^{-1}R\right) \end{pmatrix}. \tag{2.86}$$

Now we use the two identities (2.58)

$$RQ = -2P^T\,, \qquad NQ = 2\,\mathbb{1}\,, \tag{2.87}$$

in order to rewrite g:

$$g = \begin{pmatrix} -N + RN^{-1}R & -2RN^{-1} \\ -2N^{-1}R & 4N^{-1} \end{pmatrix} \tag{2.88}$$

Then it is easy to see that $g\,g' = +\mathbb{1}$, so that the right-hand side of (2.78) takes the form

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = g_N\,. \tag{2.89}$$

□

2.7 The minkowskian para-c-map in the holomorphic parametrization

For completeness and future use, we now extend the results of the previous section to the case of the second para-c-map c^{3+1}_{3+0}. We shall see later that this corresponds to the dimensional reduction of the four-dimensional minkowskian vector multiplet lagrangian along a time-like direction to three euclidean dimensions, followed by the dualization into a hypermultiplet lagrangian.

Let (M, J, g, ∇) be a special Kähler manifold and (J_1, J_2, J_3, g_N) the para-hyper-Kähler structure on $N = T^*M$ constructed in theorem 3. We denote by F the holomorphic prepotential with respect to some local system of special holomorphic coordinates (z^i) on M and by (z^i, w_i) the corresponding canonical J_1-holomorphic local coordinates of $N = T^*M \cong \wedge^{1,0}T^*M \subset T^*M \otimes \mathbb{C}$.

Proposition 6. *The expression for the para-complex structure J_2 on N in the canonical holomorphic coordinates (z^i, w_i) is the following:*

$$J_2^* dz^i = -2\sqrt{-1}\sum N^{ij}\left(d\bar{w}_j - \sqrt{-1}\bar{F}_{jkl}N^{km}(w_m - \bar{w}_m)d\bar{z}^l\right), \tag{2.90}$$

where (N^{ij}) is the inverse of the matrix (N_{ij}), with matrix elements $N_{ij} = \sqrt{-1}(F_{ij} - \bar{F}_{ij})$.

Proof. Let us first express the para-complex structure J_2 in the canonical coordinates $(x^i, y_i, \hat{x}_i, \hat{y}^i)$ associated to the ∇-affine local coordinates (x^i, y_i) on M, where now

$$z^i = x^i + \sqrt{-1}u^i \quad \text{and} \quad F_i = y_i + \sqrt{-1}v_i\,. \tag{2.91}$$

If the Kähler form of M is defined as $\omega = g(\cdot, J\cdot)$, it takes the same form

$$\omega = 2\sum dx^i \wedge dy_i \tag{2.92}$$

and, hence, (2.54) still holds. This allows us to compute the left-hand side of (2.90):

$$J_2^* dz^i = -\frac{\sqrt{-1}}{2}\frac{\partial u^i}{\partial y_j} d\hat{x}_j + \frac{1}{2}\left(\delta^i_j + \sqrt{-1}\frac{\partial u^i}{\partial x^j}\right) d\hat{y}^j\,. \tag{2.93}$$

Using the identity (2.56), which holds for special Kähler manifolds, as well as for special para-Kähler manifolds, we obtain now that

$$\delta^i_j + \sqrt{-1}\frac{\partial u^i}{\partial x^j} = -\sqrt{-1}\bar{F}_{jk}\frac{\partial u^k}{\partial y_i}\,. \tag{2.94}$$

Decomposing the last equation in real and imaginary parts yields again (2.58) and, in particular,

$$\frac{\partial u^i}{\partial y_j} = 2N^{ij}\,. \tag{2.95}$$

Using (2.93), (2.94) and (2.95) we can rewrite

$$J_2^* dz^i = -\sqrt{-1}N^{ij}(d\hat{x}_j + \bar{F}_{jk}d\hat{y}^k)\,. \tag{2.96}$$

In order to check that this coincides with the right-hand side of (2.90), let us again rewrite w_i as a function of the coordinates $(q^a, p_a) = (x^i, y_i, \hat{x}_i, \hat{y}^i)$. The real coordinates (q^a, p_a) are now related to the holomorphic coordinates (z^i, w_i) via the identification

$$T^*M \xrightarrow{\sim} \wedge^{1,0}T^*M\,, \qquad \alpha = p_a dq^a \mapsto \frac{1}{2}\left(\alpha - \sqrt{-1}J^*\alpha\right) = w_i dz^i\,. \tag{2.97}$$

Using the equations

$$J^*dx^i = -du^i\,, \quad J^*dy_i = -dv_i\,, \tag{2.98}$$

we now obtain

$$\alpha = \frac{1}{2}\hat{x}_i dz^i + \frac{1}{2}\hat{y}^i F_{ij}dz^j + \text{c.c.} \tag{2.99}$$

$$\sqrt{-1}J^*\alpha = -\left(\frac{1}{2}\hat{x}_i dz^i + \frac{1}{2}\hat{y}^i F_{ij}dz^j - \text{c.c.}\right) \tag{2.100}$$

and, thus, as before,

$$w_i dz^i = \frac{1}{2}(\alpha - \sqrt{-1}J^*\alpha) = \frac{1}{2}(\hat{x}_i + \hat{y}^j F_{ij})dz^i\,. \tag{2.101}$$

This shows, in particular, that

$$\sqrt{-1}(w_i - \bar{w}_i) = \frac{1}{2}\hat{y}^j N_{ji} \tag{2.102}$$

and, hence,

$$dw_i - \sqrt{-1}F_{ijk}N^{jl}(w_l - \bar{w}_l)dz^k = dw_i - \frac{1}{2}\hat{y}^j F_{ijk}dz^k = \frac{1}{2}(d\hat{x}_i + F_{ij}d\hat{y}^j)\,. \tag{2.103}$$

Therefore, the right-hand side of (2.90) is given by

$$-2\sqrt{-1}N^{ij}\left(d\bar{w}_j - \sqrt{-1}\bar{F}_{jkl}N^{km}(w_m - \bar{w}_m)d\bar{z}^l\right) = -\sqrt{-1}N^{ij}(d\hat{x}_j + \bar{F}_{jk}d\hat{y}^k)\,. \quad (2.104)$$

Comparing this with (2.96) yields (2.90). □

In order to compute the para-hyper-Kähler metric of N in the holomorphic coordinates (z^i, w_i), it is useful to introduce a (pseudo-)unitary co-frame

$$e^i = \sum e^i_I dz^I\,, \quad (2.105)$$

with respect to the sesquilinear (pseudo-)hermitean metric

$$h = g + e\omega = \sum -N_{IJ}dz^I d\bar{z}^J \quad (2.106)$$

on (M, J), i.e.,

$$N_{IJ} = -\sum \eta_{ij}e^i_I\bar{e}^j_J\,, \qquad (\eta_{ij}) = \mathrm{diag}(\mathbb{1}_k, -\mathbb{1}_l) \quad (2.107)$$

where (k, l) is the signature of h. We also put

$$E_i := 2\sum e^I_i\left(dw_I - \sqrt{-1}F_{IJK}N^{JL}(w_L - \bar{w}_L)dz^K\right) \qquad \text{and} \qquad E^i := \sum \eta^{ij}E_j\,, \quad (2.108)$$

where

$$(e^I_i) = (e^i_I)^{-1} \qquad \text{and} \qquad (\eta^{ij}) = (\eta_{ij})^{-1}\,. \quad (2.109)$$

Now we express the full para-hyper-Kähler structure on N in the co-frame (e^i, E^i) of $\wedge^{1,0}_{J_1}T^*N$.

Theorem 5. *Let (M, J, g, ∇) be a special Kähler manifold and (J_1, J_2, J_3, g_N) the para-hyper-Kähler structure on $N = T^*M$ constructed in theorem 3. In the co-frame (e^i, E^i), the para-hyper-Kähler structure has the following expression:*

(i) The para-hypercomplex structure $(J_1, J_2, J_3 = J_1J_2)$ is given by

$$J_1^*e^i = \sqrt{-1}e^i\,, \qquad J_1^*E^i = \sqrt{-1}E^i \quad (2.110)$$
$$J_2^*e^i = \sqrt{-1}\bar{E}^i\,, \qquad J_2^*E^i = \sqrt{-1}\bar{e}^i\,. \quad (2.111)$$

(ii) The J_1-sesquilinear (pseudo)-hermitean metric $h_N := g_N + e\omega_1$, $\omega_1 = g_N(J_1\cdot,\cdot)$, is given by

$$h_N = \sum \eta_{ij}\left(e^i\bar{e}^j - E^i\bar{E}^j\right). \quad (2.112)$$

Proof. (i) follows from proposition 6.
(ii) h_N is the unique J_1-sesquilinear form such that $\mathrm{Re}\,h_N = g_N$. Therefore it is sufficient to check that

$$g_N = \mathrm{Re}\sum \eta_{ij}\left(e^i\bar{e}^j - E^i\bar{E}^j\right). \quad (2.113)$$

In order to check this, we calculate the right-hand side in the affine coordinates. The first term is

$$
\begin{aligned}
g &= \mathrm{Re} \sum e^i \bar{e}^i && (2.114)\\
&= -\mathrm{Re}\left(N_{IJ} dz^I d\bar{z}^J\right)\\
&= -\left(N_{IJ} + \frac{\partial u^K}{\partial x^I} N_{KL} \frac{\partial u^L}{\partial x^J}\right) dx^I dx^J - 2\left(\frac{\partial u^K}{\partial x^I} N_{KL} \frac{\partial u^L}{\partial y^J}\right) dx^I dy_J -\\
&\quad -\left(\frac{\partial u^K}{\partial y^I} N_{KL} \frac{\partial u^L}{\partial y^J}\right) dy_I dy_J\,. && (2.115)
\end{aligned}
$$

The second term is

$$
\begin{aligned}
&-\mathrm{Re}\sum E_i \bar{E}_i = && (2.116)\\
&\quad = 4\mathrm{Re}\left(N^{IJ}(dw_I - iF_{IKL}N^{KM}(w_M - \overline{w}_M)dz^L)(d\overline{w}_J - i\overline{F}_{JNP}N^{NQ}(w_Q - \overline{w}_Q)d\overline{z}^P)\right)\\
&\quad = N^{IJ} d\hat{x}_I d\hat{x}^J + N^{IK} R_{KJ} d\hat{x}_I d\hat{y}^J + \frac{1}{4}(N_{IJ} + R_{IK}N^{KL}R_{LJ}) d\hat{y}^I d\hat{y}^J\,, && (2.117)
\end{aligned}
$$

where we used that (2.103). Following the same steps as for the euclidean para-c-map, the metric takes the form (2.37):

$$
g_N = \begin{pmatrix} g & 0 \\ 0 & -g^{-1} \end{pmatrix}. \qquad (2.118)
$$

□

3. Dimensional reduction of the four-dimensional vector multiplet lagrangian

In this section we obtain physical realizations of both para-c-maps by the dimensional reduction of four-dimensional vector multiplet lagrangians, which are reviewed in section 3.1. In section 3.2 and 3.3 we start with the bosonic part of the four-dimensional euclidean vector multiplet lagrangian, whose scalar target manifold is a special para-Kähler manifold M. After dimensional reduction and dualization of the gauge fields we obtain a sigma model whose target space N is seen to be para-hyper-Kähler by comparison to the results of the previous section. This gives us a physical realization of the euclidean para-c-map $c^{4+0}_{3+0} : M \to N$. In section 3.4 we discuss the reduction of the four-dimensional minkowskian vector multiplet lagrangian over time. Here the target space M of the four-dimensional theory is special Kähler,[7] while the target space N of the euclidean three-dimensional theory is again para-hyper-Kähler. Thus we obtain a physical realization of the minkowskian para-c-map $c^{3+1}_{3+0} : M \to N$.

3.1 Four-dimensional bosonic lagrangians

It was shown in [1] that the general lagrangian for $\mathcal{N} = 2$ vector multiplets can be written in a uniform way for both dimension $3+1$ (Minkowski space) and dimension $4+0$ (euclidean space). In the so-called new conventions, the bosonic part of the lagrangian takes the

[7] The calculation also applies to special pseudo-Kähler manifolds, i.e., to the case of indefinite signatures.

following form:

$$\mathcal{L}^{4+0/3+1} = -\frac{1}{2}N_{IJ}(X,\bar{X})\partial_m X^I \partial^m \bar{X}^J \\ -\frac{\hat{\iota}}{2}\left(F_{IJ}(X)F^I_{-|mn}F_-^{J|mn} - \bar{F}_{IJ}(\bar{X})F^I_{+|mn}F_+^{J|mn}\right). \tag{3.1}$$

The symbol $\hat{\iota}$ represents the para-imaginary unit e, $e^2 = 1$, in euclidean signature and the imaginary unit $i = \sqrt{-1}$ in Minkowski signature. The space-time indices $m, n, \ldots$ take the values $1, \ldots, 4$ in euclidean and $0, \ldots, 3$ in Minkowski signature, while the index $I = 1, \ldots, n$ labels n vector multiplets. The scalar fields X^I are (para-)complex, and $\bar{X}^I$ denote the (para-)complex conjugated fields. The field strength F^I_{mn} have been split into their selfdual and antiselfdual parts with respect to the Hodge-$*$-operator, according to

$$F^I_{\pm|mn} = \frac{1}{2}\left(F^I_{mn} \pm \frac{1}{\hat{\iota}}\tilde{F}^I_{mn}\right). \tag{3.2}$$

Here $\tilde{F}^I_{mn} = \frac{1}{2}\epsilon_{mnpq}F^{I|pq}$ is the dual field strength, and the convention for the ϵ-tensor is $\epsilon_{1234} = 1$ for euclidean signature and $\epsilon_{0123} = 1$ for Minkowski signature. Note that in euclidean signature the (anti)selfdual field strengths are para-complex and have eigenvalues $\pm e$ under the Hodge-$*$-operator. This non-standard definition is necessary in order that the euclidean lagrangian takes the same form as the minkowskian one.[8]

All the couplings in the lagrangian are encoded in a single (para-)holomorphic function of the scalar fields, the prepotential $F(X)$. We adopt the following standard definitions:

$$F_I = \frac{\partial}{\partial X^I}F, \qquad F_{IJ} = \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}F, \qquad \bar{F}_I = \frac{\partial}{\partial \bar{X}^I}\bar{F}, \qquad \text{etc.} \tag{3.3}$$

and

$$N_{IJ} = \hat{\iota}\left(F_{IJ} - \bar{F}_{IJ}\right), \qquad R_{IJ} = F_{IJ} + \bar{F}_{IJ}. \tag{3.4}$$

The scalar part of the lagrangian (3.1) is a sigma model, i.e., the scalar fields X^I can be interpreted as the compositions of a map X from space-time into a (para-)complex manifold M with the (para-)holomorphic coordinate maps z^I. From the lagrangian we can read off that M is equipped with a (para-)hermitean metric[9] $g = \mathrm{Re}\, h$, where

$$h = -N_{IJ}dX^I d\bar{X}^J. \tag{3.5}$$

Moreover the relations (3.3) and (3.4) imply that this metric is in addition (para-)Kähler and in fact special (para-)Kähler, because it has a (para-)Kähler potential $K(X, \bar{X})$,

$$N_{IJ} = \frac{\partial}{\partial X^I}\frac{\partial}{\partial \bar{X}^J}K(X,\bar{X}), \tag{3.6}$$

[8]In [1] we also rewrote the euclidean lagrangian and supersymmetry transformation rules in a form where all bosonic fields are real and the para-complex unit e does not appear. But then the complete analogy with the minkowskian theory is lost.

[9]For a Minkowski signature theory with standard kinetic terms, N_{IJ} is positive definite, while the metric g is negative definite. But all the results derived in the following apply to arbitrary signature.

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which is determined by the (para-)holomorphic prepotential,

$$K(X,\bar{X}) = \hat{\imath}\left(F_I \bar{X}^I - X^I \bar{F}_I\right). \tag{3.7}$$

As mentioned below Definition 3 in section 2 the existence of a (para-)holomorphic prepotential is equivalent to M being special (para-)Kähler. More details and references can be found in [1], where we also give the full lagrangian, including all fermionic terms and the supersymmetry transformation rules.

3.2 Dimensional reduction of the euclidean lagrangian

We will now perform the dimensional reduction of the euclidean lagrangian $\mathcal{L}^{4+0}$. The vector potential A^I_m of the four-dimensional field strength $F^I_{mn} = \partial_m A^I_n - \partial_n A^I_m$ decomposes into a three dimensional vector A^I_a, $a = 1,2,3$ and a scalar p^I:

$$(A^I_m) = (A^I_a, p^I := A^I_4 = A^{I|4}). \tag{3.8}$$

The resulting three-dimensional lagrangian is

$$\begin{aligned}\mathcal{L}^{4+0}_{3+0} = &-\frac{1}{2}N_{IJ}\partial_a X^I \partial_a X^J - \frac{1}{2}N_{IJ}\partial_a p^I \partial_a p^J + \\ &+\frac{1}{2}R_{IJ}\partial_a p^I \epsilon_{abc} F^J_{bc} - \frac{1}{4}N_{IJ}F^I_{ab}F^J_{ab}\,,\end{aligned} \tag{3.9}$$

where $F^I_{ab} = \partial_a A^I_b - \partial_b A^I_a$. The three-dimensional ϵ-tensor ϵ_{abc} is normalized such that $\epsilon_{123} = 1$.

In three dimensions we can dualize the n abelian vector fields A^I_a into n scalar fields s_I, so that we obtain a dual lagrangian which depends on $4n$ real scalar fields. We first introduce the dual gauge fields

$$H^I_a := \frac{1}{2}\epsilon_{abc}F^I_{bc} \Leftrightarrow F^I_{ab} = \epsilon_{abc}H^I_c\,. \tag{3.10}$$

Then we promote the Bianchi identity $\epsilon_{abc}\partial_a F^I_{bc} = 0$ of the field strength F^I_{ab} to a field equation, using Lagrange multiplier fields s_I:

$$\hat{\mathcal{L}}^{4+0}_{3+0} = \mathcal{L}^{4+0}_{3+0} + \partial_a s_I H^I_a\,. \tag{3.11}$$

As usual, the field equation for s_I is the Bianchi identity for F^I_{ab}, which can be solved by introducing the vector potential A^I_a. Plugging this back into $\hat{\mathcal{L}}^{4+0}_{3+0}$ we recover $\mathcal{L}^{4+0}_{3+0}$. But instead we can first solve the equation of motion for H^I_a, which is

$$H^I_a = N^{IJ}\partial_a s_J + N^{IJ}R_{JK}\partial_a p^K\,, \tag{3.12}$$

where N^{IJ} is the inverse of the matrix N_{IJ}. Plugging this into $\hat{\mathcal{L}}^{4+0}_{3+0}$ we obtain the dual lagrangian

$$\begin{aligned}\tilde{\mathcal{L}}^{4+0}_{3+0} = &-\frac{1}{2}N_{IJ}\partial_a X^I \partial_a \bar{X}^J - \frac{1}{2}N_{IJ}\partial_a p^I \partial_a p^J + \\ &+\frac{1}{2}R_{IK}N^{KL}R_{LJ}\partial_a p^I \partial_a p^J + \frac{1}{2}N^{IJ}\partial_a s_I \partial_a s_J + \\ &+N^{IK}R_{KJ}\partial_a s_I \partial_a p^J\,.\end{aligned} \tag{3.13}$$

The lagrangians $\mathcal{L}_{3+0}^{4+0}$ and $\tilde{\mathcal{L}}_{3+0}^{4+0}$ are dual in the sense that they give rise to equivalent field equations despite that they have a different field content. Hence, they are interpreted as two different lagrangian descriptions of the same theory. Note that they are not related by a local field redefinition. Thus the duality is not a symmetry of a given lagrangian.

3.3 The para-hyper-Kähler geometry of the reduced lagrangian

The dual lagrangian depends on $4n$ real scalar fields $\mathrm{Re}(X^I), \mathrm{Im}(X^I), p^I, s_I$. We denote the resulting target space by N, and our goal is to prove that N is (for given M) isometric to the para-hyper-Kähler manifold N constructed in section 2.

Since half of the real fields, namely those parametrizing M, combine into the para-complex fields X^I, it is natural to wonder whether one can also combine the real fields p^I, s_I, which were obtained by dimensional reduction, into para-complex fields, such that the metric of N is para-hermitean. This is indeed possible: defining n para-complex fields W_I by

$$W_I = \frac{1}{2}\left(s_I + R_{IJ}p^J + eN_{IJ}p^J\right) = \frac{1}{2}s_I + F_{IJ}p^J\,, \tag{3.14}$$

the lagrangian (3.13) takes the form

$$\begin{aligned}\tilde{\mathcal{L}}_{3+0}^{4+0} = &-\frac{1}{2}N_{IJ}\partial_a X^I \partial_a \bar{X}^J + \\ &+ 2N^{IJ}\left(\partial_a W_I - eF_{IKP}N^{KM}(W_M - \bar{W}_M)\partial_a X^P\right)\cdot \\ &\cdot \left(\partial_a \bar{W}_J - e\bar{F}_{JLQ}N^{LN}(W_N - \bar{W}_N)\partial_a \bar{X}^Q\right)\right)\,. \end{aligned}\tag{3.15}$$

From this one reads off the sesquilinear para-hermitean metric h'_N of N:

$$\begin{aligned} h'_N = &-N_{IJ}dX^I d\bar{X}^J + 4N^{IJ}\left(dW_I - eF_{IKP}N^{KM}(W_M - \bar{W}_M)dX^P\right)\cdot \\ &\cdot \left(d\bar{W}_J - e\bar{F}_{JLQ}N^{LN}(W_N - \bar{W}_N)d\bar{X}^Q\right)\,. \end{aligned}\tag{3.16}$$

Given (3.14), the verification of (3.15) is straightforward, though somewhat tedious. When performing the calculation, it is useful to note the identity

$$N^{IJ}F_{IK}\overline{F}_{JL}S^{KL} = \frac{1}{4}\left(N^{IJ}R_{IK}R_{JL} - N_{KL}\right)S^{KL}\,, \tag{3.17}$$

which holds for any symmetric matrix S^{KL}.

Since the choice (3.14) is not so obvious, let us add the following remarks. The idea is to find para-complex fields W_I such that the resulting metric is manifestly para-hermitean, i.e., the purely (anti-)para-holomorphic components vanish identically. Once we decide that s_I is contained in the real part of W_I, we know that p^J must be present in the imaginary part, and the coefficient must be adjusted in such a way that all purely (anti-)para-holomorphic terms cancel. It is easy to see that this coefficient must depend on X^I, and the obvious candidates are F_{IJ} and its real and imaginary part.

The fields X^I, W_I can be viewed as compositions of maps from space-time to N with corresponding para-holomorphic coordinate maps z^I, w_I. In comparison to section 2, the para-holomorphic coordinates (z^I, w_I) define an identification of the target space N with the para-holomorphic cotangent bundle $\wedge^{1,0}T^*M \cong T^*M$. This identification[10] defines a para-complex structure J_1' on N for which the coordinates (z^I, w_I) are para-holomorphic. Moreover we have found a metric $g_N' = \mathrm{Re}\, h_N'$ which is para-hermitean with respect to J_1'. In fact it is easy to see that g_N' is para-Kähler with respect to J_1', because

$$K = -e(F_I \bar{X}^I - \bar{F}_I X^I) - 2N^{IJ}(W_I - \bar{W}_I)(W_J - \bar{W}_J) \tag{3.18}$$

is a J_1'-para-Kähler potential:

$$h_N' = \partial\bar{\partial}K\,, \tag{3.19}$$

where

$$\partial = dX^I \frac{\partial}{\partial X^I} + dW_I \frac{\partial}{\partial W_I} \qquad \text{and} \qquad \bar{\partial} = d\bar{X}^I \frac{\partial}{\partial \bar{X}^I} + d\bar{W}_I \frac{\partial}{\partial \bar{W}_I}\,. \tag{3.20}$$

In order to show that g_N' is even para-hyper-Kähler we would have to proceed as follows: first, find a second para-complex structure J_2' which anticommutes with J_1', so that $J_3' = J_1'J_2'$ is a complex structure. This gives us an almost para-hypercomplex structure (see Definition 4 in section 2). Second, verify that J_1', J_2' are anti-isometries of g_N', while J_3' is an isometry, so that we have an almost para-hermitean-structure. Finally, check that the three corresponding fundamental two-forms are closed. By a variant of the so-called Hitchin lemma this implies that the J_α' are integrable and parallel, so that g_N' is a para-hyper-Kähler metric. Note that as the starting point for all these calculations we would have to make an educated guess for J_2', first. But now we can profit enormously from the results of section 2. By comparing (3.16) to (2.78), (2.80) and (2.82) we see that metric obtained by dimensional reduction of the lagrangian takes the form

$$g_N' = \mathrm{Re} h_N' = \mathrm{Re} \sum (e^i \overline{e}^i - E_i \overline{E}_i)\,, \tag{3.21}$$

where e^i and E_i are given by (2.71), (2.73) and (2.74), respectively. Now we can use theorem 4 in combination with remark 1. Let (M, J, g, ∇) be the special para-Kähler manifold underlying the four-dimensional lagrangian. Then (M, J, g, ∇') with $\nabla' = J \circ \nabla \circ J^{-1}$ is also a special para-Kähler manifold, to which we can associate, by theorem 4, a para-hyper-Kähler manifold (N, J_α', g_N) with metric $g_N = g \oplus g^{-1} = \mathrm{Re} \sum (e^i \overline{e}^i + E_i' \overline{E}_i')$, where the decomposition of TN is defined by ∇'. By the diffeomorphism $(X^I, W_J) \to (X^I, eW_J)$ of remark 1, this manifold is mapped to the para-hyper-Kähler manifold (N, J_α, g_N'), with metric $g_N' = g \oplus (-g)^{-1} = \mathrm{Re} \sum (e^i \overline{e}^i - E_i \overline{E}_i)$, where the decomposition of TN is defined by ∇. According to (3.16) this is the para-hyper-Kähler manifold which is associated, through dimensional reduction of the four-dimensional lagrangian, to the special para-Kähler manifold (M, J, ∇). In particular, the metric (3.16) is para-hyper-Kähler and the (para-)complex structures J_α and (para-)Kähler forms ω_α can be read off from the formulae derived in section 2. Moreover, we see that the dimensional reduction of lagrangians provides a physical realization of the euclidean para-c-map c^{4+0}_{3+0} of section 2.

[10] Equivalently, we could have used the real coordinates associated with the scalar fields ($\mathrm{Re}X^I$, $\mathrm{Re}F_I, s_I, 2p^I$) to identify the target manifold N with the cotangent bundle T^*M. This identification yields the same para-complex structure J_1' on N, namely, the canonical one induced by the para-complex structure J on M. The factor 2 in front of p^I is chosen such that the above scalar fields correspond to the coordinates $(x^I, y_I, \hat{x}_I, \hat{y}^I)$ of section 2, cf. (2.66) and (3.14).

3.4 Reduction of the Minkowski lagrangian over time

The second way to define a three-dimensional theory with signature $3+0$ is to reduce the minkowskian version of (3.1) over time. Since this is very similar to the reduction we discussed above, we only mention some key formulae.[11] The decomposition of the four-dimensional gauge field is

$$\left(A^I_m\right) = \left(A^I_a, p^I := -A^I_0 = A^{I|0}\right), \qquad a = 1,2,3\,. \tag{3.22}$$

and the reduced lagrangian is

$$\begin{aligned}\mathcal{L}^{3+1}_{3+0} = & -\frac{1}{2}N_{IJ}\partial_a X^I \partial_a X^J + \frac{1}{2}N_{IJ}\partial_a p^I \partial_a p^J - \\ & -\frac{1}{2}R_{IJ}\partial_a p^I \epsilon_{abc}F^J_{bc} - \frac{1}{4}N_{IJ}F^I_{ab}F^J_{ab}\,.\end{aligned} \tag{3.23}$$

As in section 3.2, we introduce dual gauge fields by (3.10) and promote the Bianchi identity to a field equation using a Lagrange multiplier:

$$\hat{\mathcal{L}}^{3+1}_{3+0} = \mathcal{L}^{3+1}_{3+0} - \partial_a s_I H^I_a\,. \tag{3.24}$$

Note that we took a different relative sign on the right hand side compared to (3.11). This has been done for later convenience. The prefactor of this term does not have an intrinsic meaning, as it can be compensated by rescaling the field s_I.

Integrating out the gauge fields F^I_{ab} we obtain the dual lagrangian

$$\begin{aligned}\tilde{\mathcal{L}}^{3+1}_{3+0} = & -\frac{1}{2}N_{IJ}\partial_a X^I \partial_a \bar{X}^J + \frac{1}{2}N_{IJ}\partial_a p^I \partial_a p^J + \\ & +\frac{1}{2}R_{IK}N^{KL}R_{LJ}\partial_a p^I \partial_a p^J + \frac{1}{2}N^{IJ}\partial_a s_I \partial_a s_J + \\ & + N^{IK}R_{KJ}\partial_a s_I \partial_a p^J\,.\end{aligned} \tag{3.25}$$

The structure of this lagrangian is similar to the one of (3.13) but differs in its distribution of relative signs. Since M is now complex rather than para-complex, while p^I comes from the time-like component of a gauge field, the kinetic terms of both $\mathrm{Re}(X^I)$ and $\mathrm{Im}(X^I)$ have the same sign, while p^I and s_I come with the opposite sign.[12]

As already in the original sigma model with target space M, $2n$ of the real scalars combine into the n complex scalar fields X^I. The other $2n$ real scalars p^I, s_I can be combined into n complex scalar fields by

$$W_I = \frac{1}{2}\left(s_I + R_{IJ}p^J + iN_{IJ}p^J\right) = \frac{1}{2}s_I + F_{IJ}p^J\,, \tag{3.26}$$

and by rewriting the dual lagrangian (3.25) in terms of these fields we obtain

$$\begin{aligned}\tilde{\mathcal{L}}^{3+1}_{3+0} = & -\frac{1}{2}N_{IJ}\partial_a X^I \partial_a \bar{X}^J + \\ & + 2N^{IJ}\left(\partial_a W_I - iF_{IKP}N^{KM}(W_M - \bar{W}_M)\partial_a X^P\right)\cdot \\ & \cdot\left(\partial_a \bar{W}_J - i\bar{F}_{JLQ}N^{LN}(W_N - \bar{W}_N)\partial_a \bar{X}^Q\right)\,.\end{aligned} \tag{3.27}$$

[11] We also refer to [1] for a detailed comparision between the dimensional reduction over time compared to the dimensional reduction over space.

[12] Here we assume that N_{IJ} is positive definite. The generalization to indefinite signature is straightforward.

The corresponding metric g_N on N is readily seen to coincide with the metric specified by (2.113), (2.115), (2.117), which we obtained in section 2.7 by applying the minkowskian para-c-map c^{3+1}_{3+0} to the special Kähler manifold M. In particular, g_N is para-hyper-Kähler.

Note that the para-hyper-Kähler metrics which can be obtained by the para-c-maps are not the most general para-hyper-Kähler metrics, but only a subset. This is clear from the large number of isometries. In particular, constant real shifts of the fields W_I obviously preserve the lagrangians (3.15),(3.27). More generally, looking at (3.13), (3.25), we see that constant shifts of the fields p^I, s_I are manifest symmetries of the lagrangian. Geometrically this corresponds to the fact that translations in the fiber coordinates $(\hat{x}_I, \hat{y}^I)$ are isometries of the metric (2.22), (2.37), cf. footnote 10. Physically, these isometries correspond to the gauge symmetries of the gauge fields A^I_m which have been transformed into the scalars p^I, s_I by dimensional reduction and dualization. Since it is well known that for minkowskian hypermultiplets *any* hyper-Kähler manifold is an admissible target space, we conjecture that *any* para-hyper-Kähler manifold is an admissible target space for euclidean hypermultiplets. The analogous result for vector multiplets was proven in [1], and we expect that our conjecture can be proven by similar methods. Also note that the scalar geometry of a hypermultiplet is inert under dimensional reduction, because it does not contain bosonic fields other than scalars. Therefore we can lift our lagrangian to $(4+0)$ dimensions. We expect that this extends to the fermionic part of the lagrangian in the same way as it works for minkowskian hypermultiplets [21]. Moreover one should be able to dimensionally lift the action to higher-dimensional euclidean supersymmetric actions up to the point where no euclidean supersymmetry algebra with eight real supercharges exists.

One should expect that the dimensional reduction of four-dimensional vector multiplets does not give the most general hypermultiplet manifolds. The reason is that we used a 'classical' dimensional reduction where we ignored all the massive Kaluza-Klein states. In an exact treatment one would have to integrate them out, which results in modified couplings between the lower-dimensional massless fields. Put differently, the metric which we computed is the classical approximation of the full metric, which also receives perturbative threshold corrections from integrating out massive fields. Moreover, there will also be non-perturbative corrections due to field configurations with finite action, which wind around the compact direction. Such instantons are expected to break the continuous shift symmetries which we discussed above to a discrete subgroup. Therefore we expect that after the inclusion of these corrections, the hypermultiplet manifolds are more generic than those constructed here. The investigation of such manifolds will be postponed to future work.

Following the terminology used for hyper-Kähler manifolds we will call those para-hyper-Kähler manifolds which can be obtained by one of the para-c-maps *special para-hyper-Kähler manifolds*. There is no reason to believe that if a para-hyper-Kähler manifold can be constructed by one of the para-c-maps, it can also be constructed by the other. While we have a lot of freedom in choosing M (we can pick any holomorphic or para-holomorphic function, respectively), the construction of $N = T^*M$ is then completely fixed. However, there is a subclass of the special para-hyper-Kähler manifolds which can be obtained by both para-c-maps. The reason is that we can start with a five-dimensional vector multiplet lagrangian and first reduce over time and then over space, or vice versa. As the result of the dimensional reduction should not depend on the order of steps, this gives us the desired subclass of *very special para-hyper-Kähler manifolds*. This will be the subject of section 4.

In [1] we showed that the full vector multiplet lagrangian can be written in a uniform way, which applies to both Minkowski and euclidean signature. Here we note a similar result for the bosonic part of hypermultiplet lagrangians. Indeed, using the symbol $\hat{\imath}$, the lagrangians $\tilde{\mathcal{L}}^{4+0}_{3+0}$ and $\tilde{\mathcal{L}}^{3+1}_{3+0}$ take the following form:[13]

$$\begin{aligned}\tilde{\mathcal{L}}^{4+0,\,3+1}_{3+0,\,3+0} = & -\frac{1}{2}N_{IJ}\partial_a X^I \partial_a \bar{X}^J + \\ & + 2N^{IJ}\left(\partial_a W_I - \hat{\imath}F_{IKP}N^{KM}(W_M - \bar{W}_M)\partial_a X^P\right)\cdot \\ & \cdot \left(\partial_a \bar{W}_J - \hat{\imath}\bar{F}_{JLQ}N^{LN}(W_N - \bar{W}_N)\partial_a \bar{X}^Q)\right). \end{aligned} \tag{3.28}$$

The corresponding sigma model metric $g_N = \mathrm{Re}\, h_N$ is the real part of

$$\begin{aligned} h_N = & -N_{IJ}dX^I d\bar{X}^J + 4N^{IJ}\left(dW_I - \hat{\imath}F_{IKP}N^{KM}(W_M - \bar{W}_M)dX^P\right)\cdot \\ & \cdot \left(d\bar{W}_J - \hat{\imath}\bar{F}_{JLQ}N^{LN}(W_N - \bar{W}_N)d\bar{X}^Q\right). \end{aligned} \tag{3.29}$$

The (para-)Kähler potential of g_N with respect to the (para-)complex structure J_1 is

$$K = -\hat{\imath}(F_I\bar{X}^I - \bar{F}_I X^I) - 2N^{IJ}(W_I - \bar{W}_I)(W_J - \bar{W}_J)\,, \tag{3.30}$$

with

$$h_N = \partial\bar{\partial}K \tag{3.31}$$

where

$$\partial = dX^I\frac{\partial}{\partial X^I} + dW_I\frac{\partial}{\partial W_I} \qquad \text{and} \qquad \bar{\partial} = d\bar{X}^I\frac{\partial}{\partial \bar{X}^I} + d\bar{W}_I\frac{\partial}{\partial \bar{W}_I}\,. \tag{3.32}$$

This reflects that in the framework of complex-riemannian geometry hyper-Kähler and para-hyper-Kähler geometry can be interpreted as real forms of the same complex geometry. We expect that this will be useful when extending the above result to the full hypermultiplet lagrangian.

4. Dimensional reduction of the five-dimensional vector multiplet lagrangian

In [1] we discussed the dimensional reduction of the general lagrangian for (4+1)-dimensional vector multiplets with respect to both time and space. The resulting (4+0)-dimensional and $(3+1)$-dimensional lagrangians are of the type (3.1). In both cases one can perform a further reduction to $3+0$ dimensions, and one expects that the resulting lagrangians are equivalent. We will now verify this for the bosonic parts of the lagrangians.

The bosonic fields of the five-dimensional lagrangian are n real scalars σ^I and n gauge fields A^I_μ, where $\mu = 0, \ldots, 4$. All couplings are encoded in a real prepotential $\mathcal{V}(\sigma)$ which is a general cubic polynomial in the fields σ^I. The two following expressions appear explicitly

[13] At this point the different signs in (3.11) and (3.24) turn out to be convenient.

in the lagrangian:

$$a_{IJ} = a_{IJ}(\sigma) = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\mathcal{V}(\sigma) \tag{4.1}$$

and

$$d_{IJK} = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\frac{\partial}{\partial\sigma^K}\mathcal{V}(\sigma) = \text{const.} \tag{4.2}$$

The symmetric matrix $a_{IJ}(\sigma)$ can be interpreted as the metric of a *very special real manifold* K,[14] which is parametrized by the n real scalars σ^I.

If we first reduce with respect to space, then one spatial component of each gauge field becomes a scalar, $b^I \sim A^I_4$, and combines with σ^I into a complex scalar $X^I = \sigma^I + ib^I$. The bosonic lagrangian is the minkowskian version of (3.1) with a holomorphic prepotential $F(X)$, which is determined by the real prepotential of the five-dimensional theory through

$$F(X)|_{b^I=0} = \frac{1}{2i}\mathcal{V}(\sigma)\,. \tag{4.3}$$

Thus the prepotential $F(X)$ is cubic with purely imaginary coefficients. The couplings take the special form

$$R_{IJ} = R_{IJ}(b) = d_{IJK}b^K\,, \qquad N_{IJ} = a_{IJ}(\sigma)\,. \tag{4.4}$$

The scalar manifolds M obtained this way are called *very special Kähler manifolds*, and $r^{4+1}_{3+1} : K \to M$ is called the r-map.

If we further reduce this model with respect to time, then the time-like components of the gauge fields become scalars, $p^I \sim A^I_0$. The reduced lagrangian takes the form

$$\begin{aligned}\mathcal{L}^{3+1,\ 4+1}_{3+0,\ 3+1} = {}& -\frac{1}{2}a_{IJ}\partial_a\sigma^I\partial_a\sigma^J - \frac{1}{2}a_{IJ}\partial_a b^I\partial_a b^J + \frac{1}{2}a_{IJ}\partial_a p^I\partial_a p^J - \\ & -\frac{1}{2}d_{IJK}b^K\partial_a p^I\epsilon_{abc}F^J_{bc} - \frac{1}{4}a_{IJ}F^I_{ab}F^J_{ab}\,.\end{aligned} \tag{4.5}$$

Dualizing the gauge fields into scalars we obtain a para-hyper-Kähler manifold N, which is determined by a real cubic prepotential $\mathcal{V}(\sigma)$. We call manifolds, which are obtained by the successive r-map and para-c-map, *very special para-hyper-Kähler manifolds*.

Let us now consider what happens if we perform the dimensional reductions in opposite order. We first reduce over time, and therefore the scalars b^I correspond to the time-like components of the gauge fields, $b^I \sim A^I_0$. As a consequence they combine with the σ^I into para-complex scalars $X^I = \sigma^I + eb^I$. The four-dimensional lagrangian is determined by a para-holomorphic prepotential which is cubic with purely para-imaginary coefficients:

$$F(X)\big|_{b^I=0} = \frac{1}{2e}\mathcal{V}(\sigma)\,. \tag{4.6}$$

The corresponding manifolds M are called *very special para-Kähler manifolds*. The map $r^{4+1}_{4+0} : K \to M$ is called the para-r-map.

[14]By definition, these are real manifolds with a metric defined by a real cubic polynomial, see [1].

If we now reduce over space, one space-like component of each gauge field becomes a scalar, $p^I \sim A_4^I$. The reduced lagrangian is

$$\mathcal{L}_{3+0,\;4+0}^{4+0,\;4+1} = -\frac{1}{2}a_{IJ}\partial_a\sigma^I\partial_a\sigma^J + \frac{1}{2}a_{IJ}\partial_a b^I\partial_a b^J - \frac{1}{2}a_{IJ}\partial_a p^I\partial_a p^J + \\ + \frac{1}{2}d_{IJK}b^K\partial_a p^I\epsilon_{abc}F_{bc}^J - \frac{1}{4}a_{IJ}F_{ab}^I F_{ab}^J\,. \qquad (4.7)$$

From the relations between the scalar fields b^I, p^I and the five-dimensional gauge fields it is clear that that the lagrangians (4.5) and (4.7) must be related by $b^I \leftrightarrow p^I$. To verify this we relabel the fields as indicated and perform a partial integration of the first term in the second line. Using the Bianchi identity $\epsilon_{abc}\partial_a F_{bc}^I = 0$ together with the fact that d_{IJK} is constant, we see that the lagrangians (4.5) and (4.7) agree up to a total derivative and therefore have the same equations of motion.[15] By dualizing the lagrangians $\mathcal{L}_{3+0,\,4+0}^{4+0,\,4+1}$ and $\mathcal{L}_{3+0,\,3+1}^{4+0,\,3+1}$ we can find the relation between the associated non-linear sigma models $\tilde{\mathcal{L}}_{3+0,\,4+0}^{4+0,\,4+1}$ and $\tilde{\mathcal{L}}_{3+0,\,3+1}^{4+0,\,3+1}$. The fields of the two lagrangians are related by

$$(\sigma^I, b^I, s_I, p^I) \to (\sigma^I, \pm p^I, s_I + d_{IJK}b^J p^K, \mp b^I)\,, \qquad (4.8)$$

and this defines an isometry between the corresponding very special para-hyper-Kähler manifolds. This can be summarized by

$$c_{3+0}^{4+0} \circ r_{4+0}^{4+1} = q_{3+0}^{4+1} \cong c_{3+0}^{3+1} \circ r_{3+1}^{4+1}\,, \qquad (4.9)$$

which proves proposition 4 of section 2. The corresponding commutative diagram underlies figure 1 in section 1.

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